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# ON SOME ANALYTIC PROPERTIES OF THE MODIFIED EXPONENTIAL-COSINE OPERATOR

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Let $X$ be a Banach space and $B(X)$ denote the family of bounded linear operators on $X$. Let $R^+ = [0, \infty)$. A one parameter family of operators $\{S(t) : t \in R^+\}$, $S : R^+ \to B(X)$, $S(0) = I$ (the identity operator), is called a 'modified exponential-cosine operator' if it satisfies the equation $S(t + s) + S(t - s)\ T(2s) = 2S(t)\ S(s)$, $s \leqslant t$, where $\{T(t) : t \in R^+\}$, $T : R^+ \to B(X)$, is a known semigroup of operators of class $(C_0)$. In this paper some regularity properties of $\{S(t)\}$ are studied in the strong operator topology. The sufficient conditions for measurability of $\{S(t)\}$ to imply its continuity are established. It is shown that the continuity of $\{S(t)\}$ at the origin implies its continuity everywhere. It is also shown that $S(t_1)\ S(t_2) = S(t_2)\ S(t_1)$ if and only if $S(t_1)\ T(t_2) = T(t_2)\ S(t_1)$, for all $t_1, t_2 \in R^+$. Further it is proved that if $S(t)$ is continuous, then there exist reals $M > 0$ and $\omega$ such that $\| S(t) \| \leqslant M \exp(\omega t)$. Finally some examples of the modified exponential-cosine operator and its applications to Markov processes are given.

## 1. INTRODUCTION

Let $X$ be a Banach space and $B(X)$ denote the family of bounded linear operators on $X$. Let $R^+ = [0, \infty)$ and let $I$ denote the identity operator. A one parameter family of operators $\{T(t)\ ;\ t \in R^+\}$, $T: R^+ \to B(X)$, is said to be a 'semigroup of operators' on $X$ if $T(0) = I$, and

$$T(s + t) = T(s)\ T(t),\ s,\ t \in R^+. \qquad ...(1)$$

If $T(t) f$ is continuous at $t = 0$, for any $f \in X$, it is called a semigroup of class $(C_0)$ (cf. Hille and Phillips 1957). The semigroups of operators have been extensively studied and the reader is refered to the treatise of Hille and Phillips (1957).

Sova (1968) introduced the 'cosine operator function', which is defined to be the family $\{C(t), t \in R^+\}$, $C : R^+ \to B(X)$, with $C(0) = I$ and which satisfies the functional equation

$$C(t + s) + C(t - s) = 2C(t)\ C(s),\ t, s \in R^+, t \geqslant s, \qquad ...(2)$$

(refer Kurepa 1962, Sova 1968).

As a common generalization of operator semigroups and cosine operators, Buche (1975) introduced the 'exponential-cosine operator function' $\{S(t), t \in R^+\}$, $S : R^+ \to B(X)$, $S(0) = I$, defined by

$$S(t+s) - 2S(t)\,S(s) = (S(2s) - 2S^2(s))\,S(t-s),\ s, t \in R^+, s \leqslant t. \quad \ldots(3)$$

Buche studied some boundedness and continuity properties of the exponential-cosine operator function, obtained the differential equation associated with it in the uniform operator topology, and proved, that under some conditions, $S(t) = T(t)\,C(t)$, where $\{T(t),\ t \in R^+\}$, $T : R^+ \to B(X)$, is a semigroup of operators and $\{C(t),\ t \in R^+\}$, $C : R^+ \to B(X)$, is cosine operator function.

In this paper we investigate some properties of a family of operators, which is a modified version of eqn. (3). The 'modified exponential-cosine operator function' is a one parameter family of operators $\{S(t),\ t \in R^+\}$, $S : R^+ \to B(X)$, such that

$$S(t+s) + S(t-s)\,T(2s) = 2S(t)\,S(s),\quad s, t \in R^+, s \leqslant t, \quad \ldots(4)$$

where $\{T(t)\ ;\ t \in R^+\}$, $T : R^+ \to B(X)$, is a known $(C_0)$-semigroup of operators. It may be observed that the cosine operator function becomes a particular case of the modified exponential-cosine operator function when $T(t) = I$, for all $t \in R^+$. In this paper we shall study the solution of eqn. (4) in the strong operator topology.

In section 2, we have discussed the measurability, boundedness and continuity properties of the modified exponential-cosine operator function. Some results about the commutativity of $\{S(t)\}$ and $\{T(t)\}$ are obtained in section 3.

## 2. The Measurability, Boundedness and Continuity Properties

*Proposition* 1 — Let $\{S(t),\ t \in R^+\}$, $S : R^+ \to B(X)$, be a modified exponential-cosine operator function, with $\{T(t)\}$ as the associated $(C_0)$-semigroup. Suppose $\{S(t)\}$ satisfies the inequality

$$\| S(t-s)\,S(s)f - S(t-2s)\,T(2s)f \| \leqslant K \| T(t) \| \, \| S(t-s)\,S(s)f \|, \quad \ldots(5)$$

$$t, s \in R^+,\ 2s \leqslant t,\ f \in X,$$

where $K \geqslant 0$ is some constant. Then the Lebesgue measurability of $\{S(t)f\}$ for $t > 0$ implies its continuity for $t > 0$.

Proof : Firstly, the Lebesgue measurability of $S(t)f$ for $t > 0$ implies boundedness of $S(t)f$ on every closed interval of $R^+$-$\{0\}$. This can be proved by using the functional equation (4), the assumption (5) and the fact that $\| T(t) \| \leqslant M_1 \exp(\omega t)$ for some $M_1 > 0$ and $\omega$ real. The proof follows on the same lines as that of the semigroups (ref. Hille and Phillips 1957, pp. 304-305).

The proof of the fact that the Lebesgue measurability and boundedness of $S(t)f$ implies the continuity of $S(t)f$ is also similar to that given in Hille and Phillips, (1957, pp. 305-306).

*Remark* : The inequality (5) signifies a type of internal relative boundedness of $\{S(t)\}$ with respect to $\{T(t)\}$. This inequality is satisfied in case of the examples given in section 4 of the paper.

*Definition* — Let $\{S(t),\ t \in R^+\}$, $S : R^+ \to B(X)$, be a modified exponential-cosine operator function. It is called *regular* if $\lim S(t) f = f$, as $t \to 0^+$, for every $f \in X$.

*Proposition* 2 — Let $\{S(t), t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function satisfying (4). Then there exist two non-negative constants $M$ and $\omega$ such that $\| S(t) \| \leqslant M \exp(\omega t)$, for every $t \in R^+$.

PROOF : Since $\lim_{t\to 0} S(t) f = f$, and $\lim_{t\to 0} T(t) f = f$, by the Banach Steinhaus Theorem, there exist an $\eta$, $0 < \eta \leqslant 1$ and $K \geqslant 0$ such that $0 < s \leqslant \eta$, $\| S(s) \| \leqslant K$, and $\| T(s) \| \leqslant K$. Let $m_0' = \sup_{0 \leqslant s \leqslant \eta} \| S(s) \|$, $m_0'' = \sup_{0 \leqslant s \leqslant \eta} \| T(s) \|$. Let $m_0 = \max(m_0', m_0'')$. Evidently $m_0$ is finite and bigger than or equal to one. We first prove that $\| S(ns) \| \leqslant (3m)^n$, for every $n = 1, 2, \ldots$, for all $0 \leqslant s \leqslant \eta$. We prove this by induction. The case $n = 1$ is trivial. For $n = 2$, put $t = s$ in the equation (4). Then $S(2t) = 2S^2(t) - T^2(t)$. Hence $\| S(2t) \| \leqslant 2m^2 + m^2 = 3m^2 \leqslant (3m)^2$. Now we assume that the inequality $\| S(ns) \| \leqslant (3m)^n$ is valid for all the positive integers not exceeding a fixed positive integer $n$. By the equation (4)

$$S((n+1)s) = 2\, S(ns)\, S(s) - S((n-1)s)\ T(2s),$$

$$\| S((n+1)s) \| \leqslant 2(3m_0)^n\, m_0 + (3m_0)^{n-1}\, m_0^2 \leqslant (3m_0)^{n+1}.$$

So the desired inequality is proved for all $n = 1, 2, 3, \ldots$. Further for each $t \in R^+$, there exists one and only one natural number $n \geqslant 1$ such that $(n-1)\eta < t \leqslant n\eta$. Now if we take $s = t/n$. We obtain $\| S(t) \| \leqslant (3m_0)^n = (3m_0)^{t/\eta}\,(3m_0)^{n-t/\eta} \leqslant (3m_0)\,(3m_0)^{t/\eta}$. Define $M = 3m_0$, and $\omega = (1/\eta) \log(3m_0)$. Then for each $t \in R^+$, $\| S(t) \| \leqslant M \exp(\omega t)$.

The following Corollary immediately follows from the above proposition.

*Corollary* 1 — Let $\{S(t),\ t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function satisfying the equation (4). Then there exist two non-negative constants $M$ and $\omega$ such that $\| S(t) \| \leqslant M \cosh(\omega t)$, $t \in R^+$.

*Proposition* 3 — Let $\{S(t), t \in R^+\}$ be a regular modified exponential-cosine operator function. Then the function $S(.)f$ is continuous on $R^+$ for every $f \in X$.

PROOF : We proceed indirectly and assume that there exist $f_0 \in X$ and $t_0 \in R^+$ such that $S(t) f_0$ is not continuous at the point $t_0$. Let us define for $n = 1, 2, 3, \ldots,$

$$K_n = \sup \{ \| S(t) f_0 - S(s) f_0 \|, \ | t - t_0 | \leqslant t_0/8n,$$
$$| s - t_0 | \leqslant t_0/8n, \ t, s \in R^+ \}.$$

$K_n$ is a non-increasing sequence of non-negative real numbers. Hence there exists a $K \in R^+$ such that $K_n \to K$. Our assumption of discontinuity of $S(t) f_0$ at $t_0$ implies that $K > 0$. By the definition of $K_n$, it follows that there exist two sequences $\tau_n \in R^+$, $\sigma_n \in R^+$, $\tau_n < \sigma_n$, $| \tau_n - t_0 | \leqslant t_0/8n$, $| \sigma_n - t_0 | \leqslant t_0/8n$ and $\| S(\tau_n) f_0 - S(\sigma_n) f_0 \| \geqslant K_n - 1/n$, for every $n = 1, 2, 3, \ldots$ . It is clear that $\sigma_n - \tau_n \leqslant t_0/4n$, and $\tau_n \geqslant t_0 - t_0/8n > t_0/4n$. Therefore $| 2\tau_{4n} - \sigma_{4n} - t_0 | \leqslant t_0/8n$. So by the definition of $K_n$, $\| S(\sigma_{4n}) f_0 - S(2\tau_{4n} - \sigma_{4n}) f_0 \| \leqslant K_n$, for all $n = 1, 2, \ldots$ . By (4) we have

$$2(S(t+s) - S(t)) - (S(t+s) - S(t-s)) + S(t-s)(T(2s) - I)$$
$$= 2S(t)(S(s) - I), \text{ for every } t, s \in R^+, s \leqslant t.$$

Hence

$$2 \| S(t+s) f - S(t) f \| \leqslant 2 \| S(t) \| \| S(s) f - f \|$$
$$+ \| S(t+s) f - S(t-s) f \|$$
$$+ \| S(t-s) \| \| T(2s) f - f \|,$$

for every $f \in X$, $t, s \in R^+$, $s \leqslant t$. Taking $t = \tau_{4n}$, $s = \sigma_{4n} - \tau_{4n}$ and $f = f_0$, we obtain

$$2 \| S(\sigma_{4n}) f_0 - S(\tau_{4n}) f_0 \| \leqslant 2 \| S(\tau_{4n}) \| \| S(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \|$$
$$+ \| S(\sigma_{4n}) f_0 - S(2\tau_{4n} - \sigma_{4n}) f_0 \|$$
$$+ \| S(2\tau_{4n} - \sigma_{4n}) \| \cdot \| T(2(\sigma_{4n} - \tau_{4n})) f_0 - f_0 \|.$$

Hence

$$2(K_{4n} - 1/4n) \leqslant 2 \| S(\tau_{4n}) \| \| S(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \|$$
$$+ \| S(\sigma_{4n}) f_0 - S(2\tau_{4n} - \sigma_{4n}) f_0 \|$$
$$+ \| S(2\tau_{4n} - \sigma_{4n}) \| \cdot \| T(2(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \| .$$

Hence

$$2(K_{4n} - 1/4n) \leqslant 2 \| S(\tau_{4n}) \| \| S(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \|$$
$$+ K_n + \| S(2\tau_{4n} - \sigma_{4n}) \| \| T(2(\sigma_{4n} - \tau_{4n})) f_0 - f_0 \|,$$

$$K_{4n} + (K_{4n} - K_n) \leqslant 1/2n + 2 \| S(\tau_{4n}) \| \| S(\sigma_{4n} - \tau_{4n}) f_0 - f_0 \|$$
$$+ \| S(2\tau_{4n} - \sigma_{4n}) \| \| T(2(\sigma_{4n} - \tau_{4n})) f_0 - f_0 \|.$$

Using $\| S(t) \| \leqslant M \exp(\omega t)$ and the regularity of $\{S(t)\}$ and $\{T(t)\}$, the right-hand side tends to zero as $n \to \infty$. It implies that $K_{4n} \to 0$, as $n \to \infty$, which is a contradiction. Thus we obtain the Proposition.

## 3. Commutativity of $\{S(t)\}$ and $\{T(t)\}$

*Lemma* 1 — Let $\{S(t),\ t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function, with $\{T(t)\}$ as the associated semigroup. Let $S(t)\,T(t) = T(t)\,S(t)$, $t \in R^+$. Then for each $t \in R^+$ and $n = 1, 2, 3, \ldots$, there exist $(n+1)$ constants $\alpha_{0n}, \alpha_{1n}, \ldots, \alpha_{nn}$ such that

$$S(nt) = \alpha_{0n}\,T^n(t) + \alpha_{1n}\,T^{n-1}(t)\,S(t) + \ldots + \alpha_{nn}\,S^n(t).$$

Proof : We shall use induction. For $n = 1$, the assertion is true with $\alpha_{01}=0$, $\alpha_{11} = 1$. For $n = 2$, $S(2t) = 2S^2(t) - T^2(t)$, by putting $s = t$ in the eqn. (4). Let $n \geqslant 2$, then, by the eqn. (4), $S((n+1)t) = 2S(nt)\,S(t) - S((n-1)t)\,T^2(t)$. Hence if we assume that the assertion of the Lemma is true for each $1 \leqslant k \leqslant n$, then the assertion holds for $n+1$ also, because $S(t)\,T(t) = T(t)\,S(t)$, for all $t \in R^+$.

*Proposition* 4 — Let $\{S(t), t \in R^+\}$, $S : R^+ \to B(X)$, be a regular modified exponential-cosine operator function satisfying the eqn. (4). Then $S(t_1)\,S(t_2) = S(t_2)\,S(t_1)$, for all $t_1, t_2 \in R^+$, if and only if $S(t_1)\,T(t_2) = T(t_2)\,S(t_1)$, for all $t_1, t_2 \in R^+$.

Proof : (i) *Sufficiency* — Let $S(t_1)\ T(t_2) = T(t_2)\ S(t_1)$, for all $t_1, t_2 \in R^+$. By the Lemma 1, we have $S(mt)\,S(nt) = S(nt)\,S(mt)$, $m$, $n = 1, 2, \ldots$, $t \in R^+$. Take $t_1 = r/2^k$, $t_2 = s/2^j$, $r, k, j, s$ positive natural numbers. It follows easily from above that $S(t_1)\,S(t_2) = S(t_2)\,S(t_1)$. But the rationals of the type $(r/2^k)$ are dense in $X$. In view of the continuity of $\{S(t)\}$, it follows that $S(t_1)\,S(t_2) = S(t_2)\,S(t_1)$, for all $t_1$, $t_2 \in R^+$.

(ii) *Necessity* — Let $S(t_1)\,S(t_2) = S(t_2)\ S(t_1)$, $t_1$, $t_2 \in R^+$. Using (4) with $t = s$, we obtain

$$T(t) = 2S^2(t/2) - S(t).$$

Therefore $S(t)\,T(t) = T(t)\,S(t)$, $t \in R^+$, by using the assumption. Now for any integers $m$ and $n$, $t \in R^+$, $S(mt)\,T(nt) = \sum_{i=0}^{m} \alpha_{im}\,T^{m-i}(t)\,(S(t))^i\,T^n(t)$, by the Lemma 1. Because $S(t)\,T(t) = T(t)\,S(t)$, this implies that $S(mt)\,T(nt) = T(nt)\,S(mt)$. Now the same argument as given in the sufficiency part proves the necessity part.

## 4. Examples

*Example* 1 — Let $X = C(R)$, the class of bounded uniformly continuous real-valued functions on the real line, with sup norm. Let $\{T(t)\}$ be the translation semigroup, which is defined as

$$T(t)\,f(x) = f(x + at), \qquad \ldots(6)$$

where $a > 0$ is some constant, $f \in C(R)$ and $x \in R$ (cf. Yosida 1971).

Let us define the family of bounded linear operators $\{S(t) ; t \in R^+\}$ on $X$ as follows :

$$S(t) f(x) = (1/2) (f(x + (a + b)t) + f(x + (a - b)t)), \qquad ...(7)$$

where $b$ is some constant, $0 < b < a$. It is easy to see that the family $\{S(t), t \in R^+\}$ is a regular modified exponential-cosine operator function with $\{T(t)\}$ defined by (10) as the associated semigroup, and that $\| S(t) \| \leqslant 1$, $t \in R^+$.

*Example* 2 — Let $X$ be the same Banach space as in Example 1. Let $\{T(t), t \in R^+\}$ be the $C_0$-semigroup defined as follows : For each $f \in X$, $x \in R$,

$$T(t) f(x) = \exp ( - \alpha t) f(x), \alpha > 0. \qquad ...(8)$$

Let the one-parameter family of operators $\{S(t) ; t \in R^+\}$ be defined as

$$S(t) f(x) = \exp ( - \alpha t) \sum_{k=0}^{\infty} ((\alpha t)^{2k}/(2k)!) f(x - 2k \mu), \qquad ...(9)$$

where $\mu > 0$ is some constant (cf. Buche 1971). It can be easily checked that $\{S(t) ; t \in R^+\}$ satisfies eqn. (4) with (8) as the associated semigroup.

*Example* 3 : *Mixed Semigroup* — Let $X$ be a Banach space and $B(X)$ the space of bounded linear operators on $X$. Let $\{T_1(t) ; t \in R^+\}$, $T_1 : R^+ \to B(X)$, and $\{T_2(t), t \in R^+\}$, $T_2 : R^+ \to B(X)$, be two $(C_0)$-semigroups of operators. Let $\{T_1(t)\}$ and $\{T_2(t)\}$ commute and satisfy the Trotter's (1959) condition so that

$$T(t) = T_1(t/2)\, T_2(t/2), t \in R^+ \qquad ...(10)$$

is a $(C_0)$-semigroup. The family $\{S(t) ; t \in R^+\}$, $S : R^+ \to B(X)$, defined by

$$S(t) = (T_1(t) + T_2(t))/2, t \in R^+ \qquad ...(11)$$

is a modified exponential-cosine operator function with $\{T(t)\}$ defined by (10) as the associated $(C_0)$-semigroup (Singh 1978).

*Example* 4 : *Generalized Mixed Markov Processes* — In this example the same terminology for Markov Processes as in Dynkin (1965) is followed. Consider an arbitrary state space $(E, \mathscr{B})$. Let $P(t, x, \Gamma)$ $(t \geqslant 0, x \in E, \Gamma \in \mathscr{B})$ be a Markov transition function. $\pi(t, x, \Gamma)$ $(t \geqslant 0, x \in E, \Gamma \in \mathscr{B})$ is called a 'generalized mixed Markov transition function' with $P(t, x, \Gamma)$ as the associated Markov transition function, if the following conditions are satisfied :

(i) for fixed $t$ and $x$, the function $\pi(t, x, \Gamma)$ is a measure on the $\sigma$ — algebra $\mathscr{B}$,

(ii) for fixed $t$ and $\Gamma$, $\pi(t, x, \Gamma)$ is a $\mathscr{B}$-measurable function of $x$,

(iii) $\pi(t, x, E) \leqslant 1$,

(iv) $\pi(s + t, x, \Gamma) = 2\int \pi(t, x, dy)\, \pi(s, y, \Gamma) - \int \pi(t - s, x, dy)\, P(2s, y, \Gamma)$, $t, s \geqslant 0, s \leqslant t$.

Let $X = B(E, \mathscr{B})$ be the Banach space of bounded real-valued $\mathscr{B}$-measurable functions, defined on $E$ with the natural norm $\|f\| = \sup_{x \in E} |f(x)|$. Let $P_1(t, x, \Gamma)$ and $P_2(t, x, \Gamma)$ be two Markov transition functions and $P(t, x, \Gamma)$ be the Markov transition function given by

$$P(t, x, \Gamma) = \int P_1(t/2, x, dy)\, P_2(t/2, y, \Gamma), \qquad \ldots(12)$$

$t \geqslant 0, x \in E, \Gamma \in \mathscr{B}$.

Define

$$\pi(t, x, \Gamma) = (P_1(t, x, \Gamma) + P_2(t, x, \Gamma))/2, \qquad \ldots(13)$$

$t \geqslant 0, x \in E, \Gamma \in \mathscr{B}$ (cf. Feller 1968). It can be easily checked that $\pi(t, x, \Gamma)$ defined by (13) is a generalized mixed Markov transition function, with $P(t, x, \Gamma)$ defined by (12) as the associated Markov transition function.

If $\pi(t, x, \Gamma)$ is any generalized mixed Markov transition function, with $P(t, x, \Gamma)$ as the associated Markov transition function. Then the family of operators $\{T(t), t \in R^+\}$ and $\{S(t), t \in R^+\}$ defined by

$$T(t)\, f(x) = \int P(t, x, dy) f(y), \quad f \in X, \quad x \in E,$$

$$S(t)\, f(x) = \int \pi(t, x, dy) f(y), \quad f \in X, \quad x \in E,$$

are respectively a semigroup and a modified exponential-cosine operator function with $\{T(t)\}$ as the associated semigroup.

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