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DECOMPOSITION OF NEO-RECURRENT CURVATURE TENSOR FIELD OF THE SECOND ORDER

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The decomposition of curvature tensor field in a recurrent space has been introduced by Takano (1967). In a recent paper we have decomposed the neo-recurrent curvature tensor field of the first order (Prasad and Srivastava 1978a). The object of the present paper is to decompose the neo-recurrent curvature tensor field of the second order and study some of the properties of such decomposition.

1. INTRODUCTION

Let the metric function of the Finsler space F_n be $g_{ij}(x, \dot{x})$ $(i, j = 1, 2, ..., n)$ and let us consider an other Finsler space F_m imbedded in $F_n (n > m)$ whose metric function be $g_{\alpha\beta}(u, \dot{u})$ $(\alpha, \beta = 1, 2, ..., m)$. These metric functions are such that (Rund 1959)

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})\, X^i_\alpha\, X^j_\beta\,, \; X_\alpha = \frac{\partial x^i}{\partial u^\alpha} \qquad ...(1.1)$$

The tangential component $A^\alpha (\alpha = 1, 2, ..., m)$ of the curvature vector of a curve C in F_m is given by (Chandra 1972)

$$A^\alpha = \frac{d^2 u^\alpha}{ds^2} + F^\alpha_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \qquad ...(1.2)$$

where

$$F^\alpha_{\beta\gamma}(u, \dot{u}) \overset{def}{=} P^{*\alpha}_{\beta\gamma}(u, \dot{u}) + \sum_{\nu,\tau} \bar{C}^*_{\nu\tau j}\, \overset{*}{\Omega}_{(\nu)\beta\gamma} \left(\overset{*}{t}{}^\alpha_{\tau j} - \sec \phi \tau_j \frac{du^\alpha}{ds} \right) \qquad ...(1.3)$$

is the neo-connection of F_m and $P^{*\alpha}_{\beta\gamma}$ is the induced connection (Chandra 1972).

Let the vector Y^α be the function of $(u, \dot{u})$, then the neo-covariant differentiation of Y^α with respect to u^β is defined by (Chandra 1972)

$$\overset{n}{\underset{\beta}{\nabla}} Y^\alpha = \partial_\beta Y^\alpha + \dot{\partial}_\delta Y^\alpha \partial_\beta \dot{u}^\delta + F^\alpha_{\beta\gamma} Y^\gamma \qquad ...(1.4)$$

where $\partial_\beta Y^\alpha$ and $\partial_{\dot{\delta}} Y^\alpha$ represent the partial differentiation of Y^α with respect to u^β and $\dot{u}^\delta$ respectively and $\overset{n}{\nabla}$ is the notation for the neo-covariant differentiation.

Now differentiating (1.4) neo-covariantly with respect to u^γ and commuting the indices β and γ, we get

$$\overset{n}{\underset{\beta\gamma}{\nabla}} Y^\alpha - \overset{n}{\underset{\gamma\beta}{\nabla}} Y^\alpha = N^\alpha_{\delta\beta\gamma} Y^\delta \qquad ...(1.5)$$

where

$$N^\alpha_{\delta\beta\gamma} = \Big[\Big(\frac{\partial}{\partial u^\gamma} F^\alpha_{\beta\delta} - \frac{\partial}{\partial u^\beta} F^\alpha_{\gamma\delta}\Big) + \Big(\frac{\partial}{\partial \dot{u}^\phi} F^\alpha_{\beta\delta} \frac{\partial \dot{u}^\phi}{\partial u^\gamma} - \frac{\partial}{\partial \dot{u}^\phi} F^\alpha_{\gamma\delta} \frac{\partial \dot{u}^\phi}{\partial u^\beta}\Big) + (F^\alpha_{\gamma\phi} F^\phi_{\beta\delta} - F^\alpha_{\beta\phi} F^\phi_{\gamma\delta})\Big] \qquad ...(1.6)$$

is the neo-curvature tensor field (Chandra 1972). This tensor is skew-symmetric in the indices β and γ and satisfies the relations

$$N^\alpha_{\delta\beta\gamma} + N^\alpha_{\delta\gamma\beta} = 0 \qquad ...(1.7)$$

and

$$N^\alpha_{\delta\beta\gamma} + N^\alpha_{\beta\gamma\delta} + N^\alpha_{\gamma\delta\beta} = 0. \qquad ...(1.8)$$

The Bianchi identities satisfied by the neo-curvature tensor is given by

$$\overset{n}{\underset{\phi}{\nabla}} N^\alpha_{\delta\beta\gamma} + \overset{n}{\underset{\beta}{\nabla}} N^\alpha_{\delta\gamma\phi} + \overset{n}{\underset{\gamma}{\nabla}} N^\alpha_{\delta\phi\beta} = 0. \qquad ...(1.9)$$

If $T_{\alpha\beta}(u, \dot{u})$ and $T^\alpha_\beta(u, \dot{u})$ be the tensor fields, then the commutation formulae involving the neo-curvature tensor field may be written as

$$\overset{n}{\underset{\gamma\delta}{\nabla}} T_{\alpha\beta} - \overset{n}{\underset{\delta\gamma}{\nabla}} T_{\alpha\beta} = - T_{\alpha\phi} N^\phi_{\beta\gamma\delta} - T_{\phi\beta} N^\phi_{\alpha\gamma\delta} \qquad ...(1.10)$$

and

$$\overset{n}{\underset{\gamma\delta}{\nabla}} T^\alpha_\beta - \overset{n}{\underset{\gamma\delta}{\nabla}} T^\alpha_\beta = T^\phi_\beta N^\alpha_{\phi\gamma\delta} - T^\alpha_\phi N^\phi_{\beta\gamma\delta}\,. \qquad ...(1.11)$$

The neo-recurrent curvature tensor field of the first and second order are obtained by the conditions (Prasad and Srivastava 1978b)

$$\overset{n}{\underset{\phi}{\nabla}} N^\alpha_{\delta\beta\gamma} = v_\phi N^\alpha_{\delta\beta\gamma} \qquad ...(1.12)$$

and

$$\overset{n}{\underset{\phi\theta}{\nabla}} N^{\alpha}_{\delta\beta\gamma} = a_{\phi\theta} N^{\alpha}_{\delta\beta\gamma} \qquad \text{...(1.13)}$$

where v_ϕ and $a_{\phi\theta}$ are neo-recurrence vector and tensor fields which are related by (Prasad and Srivastava (1978b)

$$a_{\phi\theta} = \overset{n}{\underset{\phi}{\nabla}} v_\theta + v_\phi v_\theta. \qquad \text{...(1.14)}$$

Using (1.12) in (1.9), we get

$$v_\phi N^{\alpha}_{\delta\beta\gamma} + v_\beta N^{\alpha}_{\delta\gamma\phi} + v_\gamma N^{\alpha}_{\delta\phi\beta} = 0. \qquad \text{...(1.15)}$$

2. Decomposition of Neo-curvature Tensor Field

Let us consider the decomposition

$$N^{\alpha}_{\delta\beta\gamma} = U^{\alpha} M_{\delta\beta\gamma} \qquad \text{...(2.1)}$$

where $M_{\delta\beta\gamma}$ is a tensor field skew-symmetric in β and γ and U^{α} is a non-zero vector field satisfying

$$U^{\alpha} v_{\alpha} = \sigma. \qquad \text{...(2.2)}$$

Using (2.1) in (1.7) and (1.8), we get respectively

$$M_{\delta\beta\gamma} + M_{\delta\gamma\beta} = 0 \qquad \text{...(2.3)}$$

and

$$M_{\delta\beta\gamma} + M_{\beta\gamma\delta} + M_{\gamma\delta\beta} = 0. \qquad \text{...(2.4)}$$

Further, decomposing the tensor field $M_{\delta\beta\gamma}$ in the form

$$M_{\delta\beta\gamma} = v_\delta P_{\beta\gamma}\,. \qquad \text{...(2.5)}$$

We shall prove the following theorems.

Theorem 2.1 — The tensor fields $M_{\delta\beta\gamma}$ and $P_{\beta\gamma}$ of (2.1) and (2.5) respectively satisfy the identities

$$a_{\phi\theta} M_{\delta\beta\gamma} + a_{\beta\theta} M_{\delta\gamma\phi} + a_{\gamma\theta}\, M_{\delta\phi\beta} = 0 \qquad \text{...(2.6)}$$

and

$$a_{\phi\theta} P_{\beta\gamma} + a_{\beta\theta} P_{\gamma\phi} + a_{\gamma\theta}\, P_{\phi\beta} = 0. \qquad \text{...(2.7)}$$

PROOF : Taking the neo-covariant differentiation of (1.9) with respect to u^{θ}, we get

$$\overset{n}{\underset{\phi\theta}{\nabla}} N^{\alpha}_{\delta\beta\gamma} + \overset{n}{\underset{\beta\theta}{\nabla}} N^{\alpha}_{\delta\gamma\phi} + \overset{n}{\underset{\gamma\theta}{\nabla}} N^{\alpha}_{\delta\phi\beta} = 0. \qquad ...(2.8)$$

Using (2.1) and (2.5) respectively in (2.8) we get (2.6) and (2.7) as U^{α} and v_{ϕ} are non-zero.

This proves the theorem.

Remark 2.1 : If the vector field U^{α} be neo-covariant constant, then the tensor fields $M_{\delta\beta\gamma}$ and $P_{\beta\gamma}$ behave like recurrence tensor fields of the first order (Prasad and Srivastava 1978a).

Theorem 2.2 — The tensor field $M_{\delta\beta\gamma}$ behave like neo-recurrent tensor field of the second order if the vector U^{α} be neo-covariant constant.

PROOF : Taking the neo-covariant differentiation of (2.1) with respect to u^{ϕ} and u^{θ} and noting the fact that U^{α} is neo-covariant constant, we get

$$\overset{n}{\underset{\phi\theta}{\nabla}} N^{\alpha}_{\delta\beta\gamma} = U^{\alpha} \overset{n}{\underset{\phi\theta}{\nabla}} M_{\delta\beta\gamma} . \qquad ...(2.9)$$

Using (1.13) and (2.1) in (2.9) and the fact that U^{α} be neo-covariant constant, we get

$$\overset{n}{\underset{\phi\theta}{\nabla}} M_{\delta\beta\gamma} = a_{\phi\theta} M_{\delta\beta\gamma} . \qquad ...(2.10)$$

This proves the statement.

Theorem 2.3 — If U^{α} be neo-covariant constant then the tensor field $P_{\beta\gamma}$ behave like neo-recurrent tensor field of the second order provided σ is constant.

PROOF : Transvecting (2.10) by U^{δ} and using (2.2) and (2.5) and the conditions of the theorem, we get

$$\overset{n}{\underset{\phi\theta}{\nabla}} P_{\beta\gamma} = a_{\phi\theta} P_{\beta\gamma} . \qquad ...(2.11)$$

This was to be shown.

Theorem 2.4 — Under the decomposition (2.1) the skew-symmetric part of recurrence tensor field $a_{\phi\theta}$ is recurrent of the first order.

PROOF : Commutating (2.10) with respect to the indices ϕ and θ we get

$$(\overset{n}{\underset{\phi\theta}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\theta\phi}{\nabla}} M_{\delta\beta\gamma}) = (a_{\phi\theta} - a_{\theta\phi}) M_{\delta\beta\gamma} . \qquad ...(2.12)$$

Using commutation formula (1.10), we get

$$-(M_{\epsilon\beta\gamma} N^{\epsilon}_{\delta\phi\theta} + M_{\delta\epsilon\gamma} N^{\epsilon}_{\beta\phi\theta} + M_{\delta\beta\epsilon} N^{\epsilon}_{\gamma\phi\theta}) = (a_{\phi\theta} - a_{\theta\phi}) M_{\delta\beta\gamma}. \qquad ...(2.13)$$

Differentiating (2.13) neo-covariantly with respect to u^{ψ}, and in the resulting equation using (1.12) and (2.13), we get

$$\overset{n}{\underset{\psi}{\nabla}} (a_{\phi\theta} - a_{\theta\phi}) = v_{\psi}(a_{\phi\theta} - a_{\theta\phi}). \qquad ...(2.14)$$

This proves the statement.

Theorem 2.5 — Under the decomposition (2.1) the relation

$$[(\overset{n}{\underset{\phi}{\nabla}} a_{\theta\psi} - \overset{n}{\underset{\psi}{\nabla}} a_{\theta\phi}) + (\overset{n}{\underset{\theta}{\nabla}} a_{\psi\phi} - \overset{n}{\underset{\phi}{\nabla}} a_{\psi\theta})$$
$$+ (\overset{n}{\underset{\psi}{\nabla}} a_{\phi\theta} - \overset{n}{\underset{\theta}{\nabla}} a_{\phi\psi}) + (a_{\phi\theta} v_{\psi} - v_{\theta} a_{\phi\psi})$$
$$+ (a_{\theta\psi} v_{\phi} - v_{\psi} a_{\theta\phi}) + (v_{\theta} a_{\psi\phi} - v_{\phi} a_{\psi\theta})] = 0 \qquad ...(2.15)$$

holds, provided the vector U^{α} be neo-covariant constant.

PROOF : Differentiating (2.10) neo-covariantly with respect to u^{ψ} and commuting the indices θ and ψ in the resulting equation, we get

$$(\overset{n}{\underset{\phi\theta\psi}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\phi\psi\theta}{\nabla}} M_{\delta\beta\gamma}) = (\overset{n}{\underset{\psi}{\nabla}} a_{\phi\theta} - \overset{n}{\underset{\theta}{\nabla}} a_{\phi\psi}) M_{\delta\beta\gamma}$$
$$+ a_{\phi\theta}(\overset{n}{\underset{\psi}{\nabla}} M_{\delta\beta\gamma}) - a_{\phi\psi}(\overset{n}{\underset{\theta}{\nabla}} M_{\delta\beta\gamma}) \qquad ...(2.16)$$

Equation (2.16) can be written as

$$[\overset{n}{\underset{\theta\psi}{\nabla}} (\overset{n}{\underset{\phi}{\nabla}} M_{\delta\beta\gamma}) - \overset{n}{\underset{\psi\theta}{\nabla}} (\overset{n}{\underset{\phi}{\nabla}} M_{\delta\beta\gamma})] = [(\overset{n}{\underset{\psi}{\nabla}} a_{\phi\theta} - \overset{n}{\underset{\theta}{\nabla}} a_{\phi\psi}) M_{\delta\beta\gamma}$$
$$+ a_{\phi\theta}(\overset{n}{\underset{\psi}{\nabla}} M_{\delta\beta\gamma}) - a_{\phi\psi}(\overset{n}{\underset{\theta}{\nabla}} M_{\delta\beta\gamma})]. \qquad ...(2.17)$$

Using the commutation formula (1.10) and the fact that $M_{\delta\beta\gamma}$ is neo-recurrent of the first order, we get

$$- v_\phi \{M_{\epsilon\beta\gamma} N^{\epsilon}_{\delta\theta\psi} + M_{\delta\epsilon\gamma} N^{\epsilon}_{\beta\theta\psi} + M_{\delta\beta\epsilon} N^{\epsilon}_{\gamma\theta\psi}\}$$

$$= [(\overset{n}{\underset{\psi}{\nabla}} a_{\phi\theta} - \overset{n}{\underset{\theta}{\nabla}} a_{\phi\psi}) + v_\psi a_{\phi\theta} - a_{\phi\psi} v_\theta] M_{\delta\beta\gamma}. \qquad \ldots(2.18)$$

Cyclic permutation of the indices ϕ, θ and ψ in (2.18) we get two more relations. On adding these three relations and simplifying the resulting expression with the help of eqn. (1.15), we get (2.16).

Theorem 2.6 — The tensor field $M_{\delta\beta\gamma}$ satisfies the relation

$$[(\overset{n}{\underset{\phi\theta\psi}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\phi\psi\theta}{\nabla}} M_{\delta\beta\gamma}) + (\overset{n}{\underset{\theta\psi\phi}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\theta\phi\psi}{\nabla}} M_{\delta\beta\gamma})$$

$$+ (\overset{n}{\underset{\psi\phi\theta}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\psi\theta\phi}{\nabla}} M_{\delta\beta\gamma})] = 0 \qquad \ldots(2.19)$$

provided that the vector U^α is neo-covariant constant.

PROOF : Cyclic permutation of the indices ϕ, θ and ψ, gives two more relations. On adding these three relations and using the fact that $M_{\delta\beta\gamma}$ is neo-recurrent of the first order, we get

$$[(\overset{n}{\underset{\phi\theta\psi}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\phi\psi\theta}{\nabla}} M_{\delta\beta\gamma}) + (\overset{n}{\underset{\theta\psi\phi}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\theta\phi\psi}{\nabla}} M_{\delta\beta\gamma})$$

$$+ (\overset{n}{\underset{\psi\phi\theta}{\nabla}} M_{\delta\beta\gamma} - \overset{n}{\underset{\psi\theta\phi}{\nabla}} M_{\delta\beta\gamma})]$$

$$= M_{\delta\beta\gamma} [(\overset{n}{\underset{\psi}{\nabla}} a_{\phi\theta} - \overset{n}{\underset{\theta}{\nabla}} a_{\phi\psi}) + (\overset{n}{\underset{\phi}{\nabla}} a_{\theta\psi} - \overset{n}{\underset{\psi}{\nabla}} a_{\theta\phi})$$

$$+ (\overset{n}{\underset{\theta}{\nabla}} a_{\psi\phi} - \overset{n}{\underset{\phi}{\nabla}} a_{\psi\theta}) + (a_{\phi\theta} v_\psi - v_\theta a_{\phi\psi})$$

$$+ (a_{\theta\psi} v_\phi - v_\psi a_{\theta\phi}) + (v_\theta a_{\psi\phi} - v_\phi a_{\psi\theta})]. \qquad \ldots(2.20)$$

In view of eqn. (2.15), eqn. (2.20) reduces to eqn. (2.19).

Theorem 2.7 — The tensor field $P_{\beta\gamma}$ satisfies the relation

$$[(\overset{n}{\underset{\phi\theta\psi}{\nabla}} P_{\beta\gamma} - \overset{n}{\underset{\phi\psi\theta}{\nabla}} P_{\beta\gamma}) + (\overset{n}{\underset{\theta\psi\phi}{\nabla}} P_{\beta\gamma} - \overset{n}{\underset{\theta\phi\psi}{\nabla}} P_{\beta\gamma})$$

$$+ (\overset{n}{\underset{\psi\phi\theta}{\nabla}} P_{\beta\gamma} - \overset{n}{\underset{\psi\theta\phi}{\nabla}} P_{\beta\gamma})] = 0 \qquad \ldots(2.21)$$

if U^α and σ be neo-covariant constant.

PROOF : Transvecting (2.19) by U^δ and using (2.5) and the fact that U^α and σ be neo-covariant constant, we get (2.21).

Theorem 2.8 — Under the decompositions (2.1) and (2.5), if the vector U^α be neo-covariant constant then the following relations hold

$$[(\overset{n}{\underset{\theta}{\nabla}} a_{\phi\psi} - \overset{n}{\underset{\psi}{\nabla}} a_{\phi\theta}) + (v_\theta a_{\phi\psi} - v_\psi a_{\phi\theta})] M_{\delta\beta\gamma}$$
$$= \sigma [M_{\phi\beta\gamma} - M_{\beta\gamma\phi} - M_{\gamma\phi\beta}] M_{\delta\theta\psi} \qquad \ldots(2.22)$$

and

$$[(\overset{n}{\underset{\theta}{\nabla}} a_{\phi\psi} - \overset{n}{\underset{\psi}{\nabla}} a_{\phi\theta}) + (v_\theta a_{\phi\psi} - v_\psi a_{\phi\theta})] M_{\delta\beta\gamma}$$
$$= \sigma [v_\phi P_{\beta\gamma} - v_\beta P_{\gamma\phi} - v_\gamma P_{\phi\beta}] P_{\theta\psi}. \qquad \ldots(2.23)$$

PROOF : Using eqns. (2.1), (2.2) and (2.5) in (2.18), we get (2.22) which yields (2.23) because of (2.5).

Theorem 2.9 — Under the decomposition (2.1), if the vector U^α be the neo-covariant constant, then for the neo-recurrent vector field satisfying,

$$\overset{n}{\underset{\theta}{\nabla}} v_\phi + v_\phi v_\theta \not\equiv 0 \qquad \ldots(2.24)$$

the tensor field $M_{\delta\beta\gamma}$ is neo-recurrent tensor field of the second order but the converse is not true in general.

PROOF : Neo-covariant differentiation of (1.12) with respect to u^θ, we get

$$\overset{n}{\underset{\phi\theta}{\nabla}} N^\alpha_{\delta\beta\gamma} = (\overset{n}{\underset{\theta}{\nabla}} v_\phi + v_\phi v_\theta) N^\alpha_{\delta\beta\gamma}. \qquad \ldots(2.25)$$

Using (2.1) in (2.25), and the fact that U^α is neo-covariant constant, we get

$$\overset{n}{\underset{\phi\theta}{\nabla}} M_{\delta\beta\gamma} = (\overset{n}{\underset{\theta}{\nabla}} v_\phi + v_\phi v_\theta) M_{\delta\beta\gamma}. \qquad \ldots(2.26)$$

In view of eqn. (1.14), eqn. (2.26) becomes,

$$\overset{n}{\underset{\phi\theta}{\nabla}} M_{\delta\beta\gamma} = a_{\phi\theta} M_{\delta\beta\gamma}. \qquad \ldots(2.27)$$

This proves the theorem.

Theorem 2.10 — Under the decompositions (2.1) and (2.5), if the vector U^α be the neo-covariant constant, then for every neo-recurrent vector field satisfying (2.24) the tensor field $P_{\beta\gamma}$ is neo-recurrent tensor field of the second order provided σ is neo-covariant constant, but the converse is not true in general.

PROOF : Transvecting (2.26) by U^δ and using (2.5) and the fact that σ is constant, we get

$$\overset{n}{\underset{\phi\theta}{\nabla}} P_{\beta\gamma} = (\overset{n}{\underset{\theta}{\nabla}} v_\phi + v_\phi v_\theta) P_{\beta\gamma} \qquad ...(2.28)$$

which in view of (1.13), becomes

$$\overset{n}{\underset{\phi\theta}{\nabla}} P_{\beta\gamma} = a_{\phi\theta} P_{\beta\gamma} \, . \qquad ...(2.29)$$

This proves the statement.

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