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SOME FIXED POINT THEOREMS IN NORMED LINEAR SPACES

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In this paper we prove some fixed point theorems for Banach type contraction mappings in normed linear spaces. The results are mainly improvements of those of Kannan (1973) and Kirk (1965). Certain properties of such mappings are defined and their mutual implications are studied.

In this paper, we improve almost every result of Kannan (1973), certain results of Belluce and Kirk (1969), Chakrabarty and Lahiri (1976), Kannan (1972) and Kirk (1965, 1971). The main tools we use are Lemma 1, the proof of which is simple and Theorem 2 which is a simple and natural consequence of Lemma 1.

NOTATION

$X =$ a normed linear space

$K =$ a nonempty closed convex subset of X

$M =$ a weakly compact subset of X, $f : K \to K$

$O(x) = \{x, fx, f^2x, \ldots\}$ is the orbit of $x \in K$

clco $A =$ the closed convex hull of A

$\delta(A) =$ the diameter of A when $A \subset X$

$\delta_0(A) = \sup_{x \in A} \| x - fx \|$ for $\phi \neq A \subset K$

$\delta_1(A) = \sup_{x,y \in A} \| x - fy \|$ for $\phi \neq A \subset K$

$\mathcal{F} = \{E \mid \phi \neq E \subset K$, E is closed, convex and $f(E) \subset E\}$, (clearly $K \in \mathcal{F}$)

$\mathcal{E} = \{E \in \mathcal{F} \mid \delta(E) > 0\}$.

For a nonempty bounded subset A of X and $x \in A$, $r_x(A) = \sup_{y \in A} \| x - y \|$, $r(A) = \inf_{y \in A} r_y(A)$ and $A_c = \{y \in A \mid r_y(A) = r(A)\}$.

We assume, throughout, that $M \cap$ clco $O(x) \neq \phi$ for each x in K.

SECTION 1

Lemma 1 — $\mathcal{F}$ has a minimal element under set inclusion and for any minimal element F of $\mathcal{F}$, clco $f(F) = F$ and hence $\delta(f(F)) = \delta(F)$.

PROOF : The first part of the theorem follows from Zorn's Lemma. Let F be a minimal element of $\mathcal{F}$ and $H = \text{clco}\, f(F)$. Then $H \subset F$ so that $f(H) \subset f(F) \subset H$. Hence $H \in \mathcal{F}$.

By the minimality of F, we have $H = F$. Hence $\delta(F) = \delta(H) = \delta(f(F))$ (since a set and its closed convex hull have the same diameter).

Theorem 2 — Suppose $\delta(f(E)) < \delta(E)$ for every E in $\mathcal{E}$. Then f has a fixed point.

PROOF : By Lemma 1, $\mathcal{F}$ has a minimal element, say, F and $\delta(f(F)) = \delta(F)$. Hence $F \notin \mathcal{E}$. Consequently, F is a singleton, say $\{x\}$ and $fx = x$.

Corollary 3 — Suppose

$$\| fx - fy \| \leqslant \max \{ \| x - fx \|, \| y - fy \| \} \ \forall\ x, y \in K$$

and $\delta_0(E) < \delta(E)$ for every E in $\mathcal{E}$. Then f has a unique fixed point.

PROOF : For $E \in \mathcal{E}$, $\delta(f(E)) \leqslant \sup_{y \in E} \| y - fy \| < \delta(E)$. Now the result follows from Theorem 2.

Corollary 4 (Kannan 1973, Theorem 1) — Suppose

$$\| fx - fy \| \leqslant \tfrac{1}{2} (\| x - fx \| + \| y - fy \|) \ \forall\ x, y \text{ in } K$$

and $\delta_0(E) < \delta(E)$ for every E in $\mathcal{E}$. Then f has a unique fixed point.

PROOF : Follows as in Corollary 3.

Corollary 5 — Suppose there exist nonnegative scalars a, b, c such that $b > 0$, $a + 2b + 2c \leqslant 1$ and

$$\| fx - fy \| \leqslant a \| x - y \| + b (\| x - fx \| + \| y - fy \|) + c (\| x - fy \| + \| y - fx \|)$$

for all x, y in K and $\delta_0(E) < \delta(E)$ for all E in $\mathcal{E}$. Then f has a fixed point.

PROOF : For E in $\mathcal{E}$,

$$\delta(f(E)) \leqslant a\, \delta(E) + 2b\, \delta_0(E) + 2c\, \delta(E) < \delta(E).$$

Now the result follows from Theorem 2.

Remark : Chakrabarty and Lahiri (1976, Theorem 1) proved the following theorem :

Let X be a reflexive Banach space and K be a nonempty closed convex bounded subset of X. Let $f : K \to K$ be such that

(i) $\| fx - fy \| \leqslant a_1 \| x - y \| + a_2 \| x - fx \| + a_3 \| y - fy \| + a_4 \| x - fy \| + a_5 \| y - fx \|$ for all x, y in K, where $a_i \geqslant 0$, $\sum_{i=1}^{5} a_i \leqslant 1$,

(ii) $\sup_{x,y \in E} \| fx - y \| < \delta(E)$ for all E in $\mathcal{E}$.

Then f has a fixed point in K. The fixed point is unique if $a_2 > 0$ or $a_3 > 0$.

According to Theorem 2, condition (ii) alone guarantees the existence of a fixed point and the uniqueness part is trivial.

Lemma 6 — Suppose K is bounded and

$$\| fx - fy \| \leqslant \max \{ \| x - fx \|, \| y - fy \| \}$$

for all x, y in K. Then, for any minimal element F in $\mathcal{F}$, $\delta_0(F) = \delta(F)$.

PROOF : Let F be a minimal element in $\mathcal{F}$. Let $x \in F$ and $U_x = \{y \in F \mid \| fx - y \| \} \leqslant \delta_0(F)\}$. Then $x \in U_x$ and U_x is a clco set. Let $y \in U_x$. Then

$$\| fx - fy \| \leqslant \max \{ \| x - fx \|, \| y - fy \| \leqslant \delta_0(F).$$

Hence $fy \in U_x$. Thus U_x is invariant under f, so that $U_x \in \mathcal{F}$. Now, from the minimality of F, $U_x = F$. Thus $x \in F \Rightarrow r_{fx}(F) \leqslant \delta_0(F)$. Define $F_0 = \{x \in F \mid r_x(F) \leqslant \delta_0(F)\}$.

Then $f(F) \subset F_0$ so that $F_0 \neq \phi$. Also F_0 is clco set. Hence $F_0 \in \mathcal{F}$. Consequently, $F_0 = F$, so that from the definition of F_0, $\delta(F) \leqslant \delta_0(F) \leqslant \delta(F)$.

The following Theorem is a generalization of Theorem 2 of Kannan (1973).

Theorem 7 — Suppose (i) K is bounded, (ii) f is continuous,

(iii) $\| fx - fy \| \leqslant \max \{ \|x - fx \|, \| y - fy \|\}$ for all x, y in K,

(iv) $\| fx - fy \| < \max \{ \|x - fx \|, \| y - fy \|\}$ if $\| x - fx \| \neq \| y - fy \|$,

(v) $E \in \mathcal{E} \Rightarrow \exists\, x \in E \ni \| x - fx \| < \delta(E)$.

Then f has a unique fixed point in K.

PROOF : Let F be a minimal element in $\mathcal{F}$. Then, by Lemma 6, $\delta_0(F) = \delta(F)$. Suppose $F \in \mathcal{E}$. Then $\exists\, x \in F \ni \| x - fx \| < \delta(F)$. Let $r = \| x - fx \|$ and $F_0 = \{y \in F \mid \| y - fy \| \leqslant r\}$. From (iii) and (iv), $f(F_0) \subset F_0$. Since f is continuous, F_0 is closed. Let H be the convex hull of $f(F_0)$. Let $z \in H$. Then $\exists\, y_1, \ldots, y_n \in F_0$ and positive scalaras $\alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $z = \sum_{i=1}^{n} \alpha_i fy_i$.

$$\exists\, i (1 \leqslant i \leqslant n) \ni \| z - fz \| \leqslant \| y_i - fy_i \|. \qquad \ldots(1)$$

If not, $\| z - fz \| > \| y_i - fy_i \| \ \forall\ i$ so that, from (iv), $\| fy_i - fz \| < \| z - fz \| \ \forall\ i$. Now

$$\| z - fz \| = \| \sum_{i=1}^{n} \alpha_i fy_i - \sum_{i=1}^{n} \alpha_i fz \| \leqslant \sum_{i=1}^{n} \alpha_i \| fy_i - fz \|$$

$$< \sum_{i=1}^{n} \alpha_i \| z - fz \|$$

$$= \| z - fz \|$$

which is a contradiction.

From (1), $\| z - fz \| \leqslant r$ so that $z \in F_0$. Hence $H \in F_0$.

Consequently $f(H) \subset f(F_0) \subset H$. Since f is continuous, it follows that $f(\bar{H}) \subset \bar{H}$.

Hence $\bar{H} \in \mathcal{F}$. But $\bar{H} \subset F$. Hence, by minimality of F, $\bar{H} = F$.

But, since $H \subset F_0$ and F_0 is closed, $\bar{H} \subset F_0$.

Thus $F \subset F_0$ which is a contradiction.

Hence $F \notin \mathcal{E}$. Consequently, f has a fixed point in K.

The uniqueness follows from (iii).

Remarks: (1) In Theorem 7, (v) can be replaced by (v′) K has normal structure (see Definition 1 in Section 2).

(2) In Theorem 7, continuity of f cannot be dropped, in view of the following example.

Example 1 — Let $X = \mathrm{R}$, $K = [0, 1]$, $f: K \to K$ be given by

$$fx = \begin{cases} x/2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\mathcal{F} = \{K\}$, and f satisfies all the conditions of Theorem 7 except (ii). Clearly f has no fixed point.

The following example shows that Theorem 7 is a proper generalization of Theorem 2 of Kannan (1973).

Example 2 — Let $X = \mathrm{R}$, $K = [0, 1]$, $f: K \to K$ be defined by

$$fx = \tfrac{2}{5} x \ \forall\ x.$$

Lemma 8 — Suppose K is bounded and

$$\| fx - fy \| \leqslant \max \{ \| x - y \|, \| x - fy \|, \| y - fx \|, \min \{ \| x - fx \|, \| y - fy \| \} \} \quad \ldots(2)$$

for all x, y in K. If F is a minimal element in $\mathcal{F}$, then F_c is a nonempty closed convex set.

PROOF: Let $A_n = \{ x \in F \mid r_x(F) \leqslant r(F) + (1/n)\}$.

Clearly A_n is a nonempty closed convex set.

Let $x \in A_n$ and $U_x = \{y \in F \mid \| fx - y \| \leqslant r(F) + (1/n)\}$.

Then $x \in U_x$ and U_x is closed and convex.

Let $y \in U_x$. Then, from (2), $\| fx - fy \| \leqslant r(F) + (1/n)$. Hence $fy \in U_x$ so that $U_x \in \mathcal{F}$.

By the minimality of F, we have $U_x = F$ so that $r_{fx}(F) \leqslant r(F) + (1/n)$. Hence $A_n \in \mathcal{F}$. Thus $A_n \cap M$ is a nonempty weakly compact set. Now

$$F_c = \bigcap_{n=1}^{\infty} A_n \supset \bigcap_{n=1}^{\infty} (A_n \cap M) \neq \phi.$$

The following theorem is a generalization of Kirk's Theorem (1965).

Theorem 9 — Suppose K is bounded, $r(E) < \delta(E)$ for each E in $\mathcal{E}$ and f satisfies (2). Then f has a fixed point in K.

PROOF: Let F be a minimal element in $\mathcal{F}$. By Lemma 8, F_c is a nonempty closed convex set. We show that F_c is f-invariant. Let $x \in F_c$ and $U_x = \{y \in F \mid \| fx - y \| \leqslant r(F)\}$. Then $x \in U_x$ and U_x is a closed convex set.

By hypothesis, it follows that $f(U_x) \subset U_x$ so that $U_x \in \mathcal{F}$. By the minimality of F, we have $U_x = F$. Hence $fx \in F_c$. Thus $F_c \in \mathcal{F}$. By the minimality of F, again, $F = F_c$. Hence by hypothesis, $F \notin \mathcal{E}$. Hence F is a singleton, say $\{x\}$, and $fx = x$.

Remark: Example 1 shows that the above Theorem fails if min. in (2) is dropped.

Theorem 10 — Suppose K is bounded and

$$\| fx - fy \| \leqslant \max \{ \| x - y \|, \| x - fx \|, \| y - fy \|, \| x - fy \|, \| y - fx \| \} \quad \ldots(3)$$

for all x, y in K. If F is a minimal element of $\mathcal{F}$, then

$$r(F) < \delta_0(F) \Rightarrow \delta_0(F) = \delta(F).$$

PROOF: Let $F_0 = \{x \in F \mid r_x(F) \leqslant \delta_0(F)\}$. Then F_0 is a nonempty closed convex set. Let $x \in F_0$ and

$$U_x = \{y \in F \mid \| fx - y \| \leqslant \delta_0(F)\}.$$

Then $x \in U_x$ and U_x is a closed convex set.

Let $y \in U_x$. Then, using (3), we get

$$\| fx - fy \| \leqslant \max \{ r_x(F), \| y - fy \|, \| y - fx \| \} \leqslant \delta_0(F).$$

Hence $fy \in U_x$ so that U_x is f-invariant and hence $U_x \in \mathcal{F}$.

By the minimality of F, we have $F = U_x$. Hence $r_{fx}(F) \leqslant \delta_0(F)$ so that, $F_0 \in \mathcal{F}$. By the minimality of F, we have $F_0 = F$.

Hence $\delta(F) \leqslant \delta_0(F)$. Consequently $\delta(F) = \delta_0(F)$.

Theorem 11 — Suppose K is bounded, $\delta_0(E) < \delta(E)$ and $r(E) < \delta(E)$ for all E in $\mathcal{E}$ and f satisfies (3). Then f has a fixed point in K.

PROOF : Let F be a minimal element of $\mathcal{F}$. Suppose $F \in \mathcal{E}$. Since, by Theorem 10, $r(F) < \delta_0(F) \Rightarrow \delta_0(F) = \delta(F)$, we must have $r(F) \geqslant \delta_0(F)$.

As in the proof of Lemma 8, it can now be shown that F_c is a nonempty closed convex set.

Now, proceeding as in Theorem 9, we get the result.

Remark : Example 1 shows that, in Theorem 11, the condition "$\delta_0(E) < \delta(E)$ for all E in $\mathcal{E}$" cannot be dropped.

The following theorem is a generalization of Theorems 2 and 3 of Belluce and Kirk (1969) and Theorem 6 of Kannan (1973).

Theorem 12 — Suppose (i) K is bounded, (ii) f is continuous,

(iii) $\| fx - fy \| \leqslant \max \{ \| x - y \|, \min (\| x - fy \|, \| y - fx \|) \}$ for all x, y in K, and

(iv) $E \in \mathcal{E} \Rightarrow \exists\, x \in E \ni \limsup_{n \to \infty} \| x - f^n x \| \leqslant \delta(E)$.

Then f has a fixed point.

PROOF : Let F be a minimal element of $\mathcal{F}$. Let $x \in F$ and $r_m = \sup_{n \geqslant m} \| x - f^n x \|$. Fix m.

Let $P = \{ y \in F \mid \| y - f^n x \| \leqslant r_m$ for all sufficiently large $n \}$.

Then $x \in P$, P is convex and $f(P) \subset P$.

Since f is continuous, it follows that $f(\bar{P}) \subset \bar{P}$.

Hence, by the minimality of F, $\bar{P} = F$.

Let $D = \bigcap_{n=1}^{\infty}$ clco $(O(f^n x))$. Then D is a nonempty subset of F.

Let $F_0 = \{y \in F \mid r_y(F) \leqslant r_m\}$. Clearly F_0 is closed and convex. Since $\bar{P} = F$, it follows that $D \subset F_0$ so that $F_0 \neq \phi$.

Let $y \in F_0$. Then, by (iii), $\| fy - fz \| \leqslant r_m \; \forall \; z \in F$ and hence, since $F = \text{clco} \; f(F)$, it follows that $r_{fy}(F) \leqslant r_m$. Thus $F_0 \in \mathcal{F}$ and by the minimality of F, $F_0 = F$. Hence $\delta(F) \leqslant r_m$. This being true for each positive integer m, follows that $\delta(F) \leqslant \limsup\limits_{n \to \infty} \| x - f^n x \|$. Hence by (iv), $F \notin \mathcal{E}$.

The following theorem is a generalization of Theorem 2.1 of Kirk (1971) and can be proved on the same lines.

Theorem 13 — Suppose P is a nonempty weakly compact convex bounded subset of a closed convex subset Q of X, P has normal structure, and $g : P \to Q$ is such that

$$\| gx - gy \| \leqslant \max \{ \| x - y \|, \min (\|x - gy \|, \| y - gx \|)\}$$

for all x, y in P and $g(\partial_Q(P)) \subset P$.

Then g has a fixed point in P.

Section 2

In this section, we study mutual implications among certain properties of f. We start with

Definition 1 (Brodskii and Milman 1948) — A nonempty bounded closed convex subset F of X is said to have normal structure (n.s) if each closed convex subset E (of F) containing more than one point, contains a point x such that $r_x(E) < \delta(E)$. f is said to have property

(A) if $\| fx - fy \| \leqslant \frac{1}{2} (\| x - fx \| + \| y - fy \|)$ for all x, y in K.

(B) if $E \in \mathcal{E} \Rightarrow \exists \; x \in E \ni \| x - fx \| < \delta_0(E)$,

(C) if $E \in \mathcal{E} \Rightarrow \exists \; x \in E \ni \sup\limits_n \| x - f^n x \| < \delta_1(E)$,

(A') if (i) $\|fx - fy \| \leqslant \max (\| x - fx \|, \| y - fy \|)$ for all x, y in K, and (ii) $\|fx - fy\| < \max(\| x - fx \|, \|y - fy\|)$ if $\| x - fx \| \neq \|y - fy \|$,

(A'') if (i) in (A') holds,

(B') if $E \in \mathcal{E} \Rightarrow \exists \; x \in E \ni \| x - fx \| < \delta(E)$,

(C') if $E \in \mathcal{E} \Rightarrow \exists \; x \in E \ni \sup\limits_n \| x - f^n x \| < \delta(E)$, and

(C'') if $E \in \mathcal{E} \Rightarrow \exists \; x \in E \ni \limsup\limits_{n \to \infty} \| x - f^n x \| < \delta(E)$.

Properties (A), (B) and (C) are due to Kannan (1973). We observe that if K has (n.s.), then f has (B') and (C').

The following implications are trivial :

$$(A) \Rightarrow (A') \Rightarrow (A''); (B) \Rightarrow (B') \text{ and } (C) \Rightarrow (C') \Rightarrow (C'').$$

Theorem 14 — $(C') \Rightarrow (C)$ and hence $(C') \Leftrightarrow (C)$.

PROOF : Let $E \in \mathcal{E}$. Then $\delta_1(E) > 0$. By Lemma 1, $\mathcal{F} \cap E (= \{D \in \mathcal{F} \mid D \subset E\})$ has a minimal element, say, F and $\delta(F) = \delta(f(F))$. If F is a singleton set, say $\{x\}$, then $fx = x$ so that $\sup_n \| x - f^n x \| = 0 < \delta_1(E)$.

Suppose $F \in \mathcal{E}$. Since f has (C'), $\exists\, x \in F \ni \sup_n \| x - f^n x \| < \delta(F)$. But $\delta(F) = \delta(f(F)) \leqslant \delta_1(E)$ so that $\sup_n \| x - f^n x \| < \delta_1(E)$. Thus f has (C).

Corollary 15 — K has (n.s) $\Rightarrow f$ has (C).

Remark : Kannan (1973, Proposition 2) proved the above result under the assumption that f is non-expansive.

Theorem 16 — Suppose f has (A''). Then $(B') \Rightarrow (B)$ and hence $(B') \Leftrightarrow (B)$.

PROOF : Similar to that of Theorem 14.

Corollary 17 — Suppose f has (A'). Then f has $(B) \Leftrightarrow f$ has (B').

Corollary 18 (Kannan 1972) — Suppose K has (n.s) and f has (A). Then f has (B).

Theorem 19 — Suppose f has (A'). Then f has $(B) \Leftrightarrow f$ has (C).

PROOF : Suppose f has (B). Let $E \in \mathcal{E}$. Then $\delta_1(E) > 0$. By Lemma 1, $\mathcal{F} \cap E$ has a minimal element F and $\delta(F) = \delta(f(F))$. If F is singleton $\{x\}$, then $\sup_n \| x - f^n x \| = 0 < \delta_1(E)$. Suppose $F \in \mathcal{E}$. Since f has (B), $\exists\, x \in F \ni \| x - fx \| < \delta_0(F)$. Since f has (A'), $\{\| f^n x - f^{n+1} x \|\}$ is decreasing and hence, for each n, $\| fx - f^{n+1} x \| \leqslant \| x - fx \| < \delta_0(F) \leqslant \delta(F) = \delta(f(F)) \leqslant \delta_1 (E)$ so that f has (C).

Let $E \in \mathcal{E}$. Then $\delta_0(E) > 0$ since f has atmost one fixed point. If $F = \{x\}$, then $\| x - fx \| = 0 < \delta_0(E)$. Suppose $F \in \mathcal{E}$. Since f has (C), there exists $x \in F$ such that

$$\begin{aligned} \sup_n \| x - f^n x \| < \delta_1(F) \leqslant \delta(F) &= \delta(f(F)) \\ &\leqslant \delta_0(F) \text{ (since } f \text{ has } (A')) \\ &\leqslant \delta_0(E). \end{aligned}$$

Thus f has (B).

Corollary 20 (Kannan 1973, Proposition 1) — Suppose f has (A). Then f has $(B) \Leftrightarrow f$ has (C).

Theorem 21 — Suppose f has (A'). Then (B), (C), (B') and (C') are equivalent.

PROOF : It follows from Theorems 14, 16 and 19.

Note : (A) and (A') are not equivalent even in the presence of continuity (see Example 2).

Wong (1974) proved that if $f: [0, 1] \to [0, 1]$ has (A) and 0, 1 are in the range of f, then $f(\frac{1}{2}) = \frac{1}{2}$. But this result fails if (A) is replaced by (A'). In fact, in this case, f may not have even a fixed point (see Example 3).

Example 3 — $f: [0, 1] \to [0, 1]$ is defined by

$$fx = \begin{cases} 1 & \text{if } x = 0 \\ x/2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

The following example shows that (A') and (A'') are not equivalent.

Example 4 — $f: [0, 1] \to [0, 1]$ is defined by

$fx = x/2$ for all x.

Definition 2 — (Belluce and Kirk 1969) f is said to have diminishing orbital diameters if $x \in K$ and $\delta(O(x)) > 0$ imply $\lim\limits_{n \to \infty} \delta(O(f^n x)) < \delta(O(x))$.

Theorem 22 — If f has diminishing orbital diameters, then f has (C') and Hence (C).

PROOF : Let $E \in \mathcal{E}$ and $x \in E$.

If $\delta(O(x)) = 0$, then $\sup\limits_n \| x - f^n x \| = 0 < \delta(E)$.

If $\delta(O(x)) > 0$, then, by hypothesis, there exists a positive integer N such that $\delta(O(f^N x)) < \delta(O(x))$.

Hence $\sup\limits_n \| f^N x - f^n(f^N x) \| \leqslant \delta(O(f^N x)) < \delta(O(x)) \leqslant \delta(E)$.

Hence f has (C').

Remark : Kannan (1973, Proposition 3) proved that f has (C) under the additional hypothesis that f is non-expansive.

We conclude this paper with two questions:

Question 1 : Are the properties (C') and (C'') equivalent ? (Of course, in the presence of (i), (ii) and (iii) of Theorem 12, they are equivalent).

In view of Theorems 7 and 21 the following question is interesting.

Question 2 : Does f have a fixed point if (i) K is bounded (ii) f is continuous (iii) f has (A') and (iv) f has (C'') ?

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