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STRESS ANALYSIS IN BONDED SECTOR PLATES

A. K. SINGHAL AND B. S. RAMACHANDRA RAO

Department of Mathematics, Indian Institute of Technology, Powai, Bombay 400076

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A two-dimensional elastostatic boundary value problem associated with two isotropic, homogeneous bonded sector plates is solved by developing the eigenfunctions in angular (circular) coordinate. Numerical convergence of these functions at the circular bondary for some prescribed boundary conditions is presented and the behaviour of normal stress along the bonded interface is shown graphically.

INTRODUCTION

The wide spread use of composite materials in many engineering structures has motivated researchers to investigate solutions of boundary value problems in linear theory of elasticity for various geometries involving two or more materials. Composite materials provide a combination of properties which are not present in any one of the materials acting alone. To use composite/bonded structures in designs, structural engineers need mathematical analysis of a model that adequately predicts their behaviour under various loading conditions. The mathematical analysis of some of these problems has attracted the attention of Iyengar (1961), Iyengar and Alwar (1963), Dundurs (1967, 1969), Bogy (1968, 1970, 1971, 1976), Lo and Conway (1974, 1975, 1976), Rao (1971), Hess (1969a, b), Choi and Horgan (1978) and others.

In this paper, the stress analysis of a two-dimensional elastostatic boundary value problem associated with bonded sector plates has been presented. The sector plate is formed by bonding two sector plates of equal radii along a common radial edge. The two plates are made up of elastically different materials and can be of arbitrary angular span. The bond between the plates is assumed to be perfect which requires that the stresses and displacements across the interface must be continuous. A state of plane stress or plane strain is assumed so that the problem is treated as two-dimensional. Eigenfunctions of the biharmonic equation in plane polar coordinates is developed for the stress-free boundary conditions on the radial edges. Numerical results are presented when an arbitrary, otherwise self-equilibrating load is prescribed on the circular edge of the sector plate.

The analysis consists of two parts. In the first part, the general expressions for stresses and displacements are developed for a homogeneous, isotropic, elastic sector plate. In the second part, two such plates are considered to be bonded together along a common radial edge and the conditions are derived for which the assumed stress field exists in each plate.

Homogeneous Sector Plate

Consider a homogeneous, isotropic, elastic sector plate, plate occupying the region $0 \leqslant r \leqslant a$, $-\delta \leqslant \theta \leqslant \delta$ (see Fig. 1). A state of plane stress or plane strain

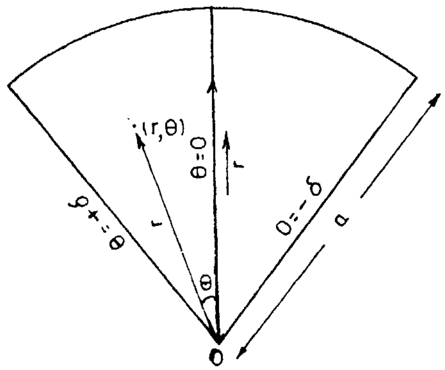

Fig. 1. Homogeneous sector plate.

will be assumed, so that the stresses may be defined in terms of the biharmonic function $\phi(r, \theta)$, viz.,

$$\triangle \triangle \phi = 0 \qquad \text{...(1)}$$

where the Laplacian operator $\triangle$ in plane polar coordinates is given by

$$\triangle \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The expressions for stresses in terms of the function $\phi(r, \theta)$ and its derivatives are given by

$$\left.\begin{array}{ll} \text{(a)} \quad \sigma_r = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}, & \text{(b)} \quad \sigma_\theta = \dfrac{\partial^2 \phi}{\partial r^2} \\ \text{(c)} \quad \tau = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta}\right) & \end{array}\right\} \qquad \text{...(2)}$$

where σ_r, σ_θ are the normal stresses and τ the shear stress.

We seek the solution of (1) in the form

$$\phi(r, \theta) = (r/a)^{(s+\lambda)/s} F(\theta) \qquad \text{...(3)}$$

λ being an eigenvalue. Substitution of (3) into (1) gives a fourth order ordinary differential equation in $F(\theta)$, viz.,

$$F^{iv} + [(p+1)^2 + (p-1)^2]\, F'' + [(p+1)^2 (p-1)^2]\, F = 0 \qquad \ldots(4)$$

where $p = \lambda/\delta$.

Equation (4) has its solution in the form

$$F(\theta) = a_1 \cos (p-1)\,\theta + a_2 \sin (p-1)\,\theta + a_3 \cos (p+1)\,\theta + a_4 \sin (p+1)\,\theta. \qquad \ldots(5)$$

Thus, the general expression for the solution of $\phi(r, \theta)$ can be written as follows :

$$\phi(r, \theta) = (r/a)^{(p+1)} [a_1 \cos (p-1)\,\theta + a_2 \sin (p-1)\,\theta + a_3 \cos (p+1)\,\theta + a_4 \sin (p+1)\,\theta]. \qquad \ldots(6)$$

Using eqns. (2), the expressions for the stress components may be written in the following form

$$\sigma_r = (r/a)^{(p-1)} [-p(p-3)\{a_1 \cos (p-1)\,\theta + a_2 \sin (p-1)\,\theta\} - p(p+1)\{a_3 \cos (p+1)\,\theta + a_4 \sin (p+1)\,\theta\}] \qquad \ldots(7a)$$

$$\sigma_\theta = p(p+1)(r/a)^{(p-1)} [a_1 \cos (p-1)\,\theta + a_2 \sin (p-1)\,\theta + a_3 \cos (p+1)\,\theta + a_4 \sin (p+1)\,\theta] \qquad \ldots(7b)$$

$$\tau = p(r/a)^{(p-1)} [(p-1)\{a_1 \sin (p-1)\,\theta - a_2 \cos (p-1)\,\theta\} + (p+1)\{a_3 \sin (p+1)\,\theta - a_4 \cos (p+1)\,\theta\}]. \qquad \ldots(7c)$$

From (7*b*, *c*), the stresses on the radial edges of the sector plate will be given as follows :

$$\left.\begin{aligned} \sigma_\theta &= p(p+1)\, A(r/a)^{(p-1)} \\ \tau &= pC(r/a)^{(p-1)} \end{aligned}\right\} \text{ on } \theta = -\delta \qquad \ldots(8a)$$

$$\left.\begin{aligned} \sigma_\theta &= p(p+1)\, B(r/a)^{(p-1)} \\ \tau &= pD(r/a)^{(p-1)} \end{aligned}\right\} \text{ on } \theta = +\delta \qquad \ldots(8b)$$

where A, B, C, D are constants. Substitution of (8) into (7) will give four linear simultaneous equations in a_1, a_2, a_3 and a_4 which are solved in terms of A, B, C, D and p giving

$$\begin{aligned} a_1 &= [(A+B)(p+1) \sin (p+1)\,\delta + (C-D) \cos (p+1)\,\delta]/S^+ \\ a_2 &= -[(A-B)(p+1) \cos (p+1)\,\delta - (C+D) \sin (p+1)\,\delta]/S^- \\ a_3 &= -[(A+B)(p-1) \sin (p-1)\,\delta + (C-D) \cos (p-1)\,\delta]/S^+ \\ a_4 &= [(A-B)(p-1) \cos (p-1)\,\delta - (C+D) \sin (p-1)\,\delta]/S^- \end{aligned} \qquad \ldots(9)$$

where $S^+ = 2(\sin 2\lambda + p \sin 2\delta)$ and $S^- = 2(\sin 2\lambda - p \sin 2\delta)$.

Upon substituting for a_1, a_2, a_3 and a_4 from (9) into (6) and simplifying, $\phi(r, \theta)$ can be written in the form

$$\begin{aligned}\phi(r, \theta) = (r/a)^{(p+1)} \frac{1}{2d} [&(p+1) g_1 \sin\{(p+1)\delta - (p-1)\theta\} \\ &- (p-1) g_1 \sin\{(p-1)\delta - (p+1)\theta\} \\ &- g_4 \cos\{(p+1)\delta + (p-1)\theta\} \\ &+ (p+1) g_2 \sin\{(p+1)\delta + (p-1)\theta\} \\ &+ g_3 \cos\{(p+1)\delta - (p-1)\theta\} \\ &- (p-1) g_2 \sin\{(p-1)\delta + (p+1)\theta\} \\ &- g_3 \cos\{(p-1)\delta - (p+1)\theta\} \\ &+ g_4 \cos\{(p-1)\delta + (p+1)\theta\}]\end{aligned} \quad \text{...(10a)}$$

where

$$d = \sin^2 2\lambda - p^2 \sin^2 2\delta \quad \text{...(10b)}$$

$$\left.\begin{aligned} g_1 &= A \sin 2\lambda - Bp \sin 2\delta, \; g_2 = B \sin 2\lambda - Ap \sin 2\delta \\ g_3 &= C \sin 2\lambda + Dp \sin 2\delta, \; g_4 = D \sin 2\lambda + Cp \sin 2\delta. \end{aligned}\right\} \quad \text{...(10c)}$$

Coker and Filon (1957) have shown that the displacements in the sector plate are given by the relations

$$Eu = \alpha r \frac{\partial \psi}{\partial \theta} - (1 + \mu) \frac{\partial \phi}{\partial r} \quad \text{...(11a)}$$

$$Ev = \alpha r^2 \frac{\partial \psi}{\partial r} - (1 + \mu) \frac{1}{r} \frac{\partial \phi}{\partial \theta} \quad \text{...(11b)}$$

where $\alpha = 1$, plane stress; $\alpha = 1 - \mu^2$, plane strain; u and v are the displacements in radial and tangential directions respectively; E is the Young's modulus; μ the Poisson's ratio; and $\psi(r, \theta)$ an auxiliary displacement function defined by

$$\Delta\psi = 0, \; \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial \theta}\right) = \Delta\phi. \quad \text{...(12)}$$

A function $\psi(r, \theta)$ satisfying (12) is

$$\begin{aligned}\psi(r, \theta) = \frac{2}{d(p-1)} (r/a)^{(p-1)} [&- g_3 \sin\{(p+1)\delta - (p-1)\theta\} \\ &+ (p+1) g_1 \cos\{(p+1)\delta - (p-1)\theta\} \\ &- g_4 \sin\{(p+1)\delta + (p-1)\theta\} \\ &- (p+1) g_2 \cos\{(p+1)\delta + (p-1)\theta\}].\end{aligned} \quad \text{...(13)}$$

The quantities of immediate interest are the displacements on the radial edges of the sector plate (i.e., $\theta = \pm\,\delta$). These are obtained from (11), (13) and (10) in the form given below :

$$\theta = -\,\delta \left\{ \begin{aligned} Eu &= (r/a)^p\,[(2\alpha/d)\,\{(p+1)\,(g_1 \sin 2\lambda + g_2 \sin 2\delta) \\ &\qquad + g_3 \cos 2\lambda - g_4 \cos 2\delta\} - (p+1)(1+\mu)\,A] \\ Ev &= (r/a)^p\,[(2\alpha/d)\,\{(p+1)(g_1 \cos 2\lambda - g_2 \cos 2\delta) \\ &\qquad - g_3 \sin 2\lambda - g_4 \sin 2\delta\} + (1+\mu)\,C] \end{aligned} \right. \qquad \ldots(14a)$$

$$\theta = +\,\delta \left\{ \begin{aligned} Eu &= (r/a)^p\,[(2\alpha/d)\,\{(p+1)\,(g_1 \sin 2\delta + g_2 \sin 2\lambda) \\ &\qquad + g_3 \cos 2\delta - g_4 \cos 2\lambda\} - (p+1)\,(1+\mu)\,B] \\ Ev &= (r/a)^p\,[(2\alpha/d)\,\{(p+1)(g_1 \cos 2\delta - g_2 \cos 2\lambda) \\ &\qquad - g_3 \sin 2\delta - g_4 \sin 2\lambda\} + (1+\mu)\,D] \end{aligned} \right. \qquad \ldots(14b)$$

Bonded Sector Plate

Let us consider a bonded sector plate made up of two homogeneous sector plates of equal radii and bonded along a common radial edge (see Fig. 2). We

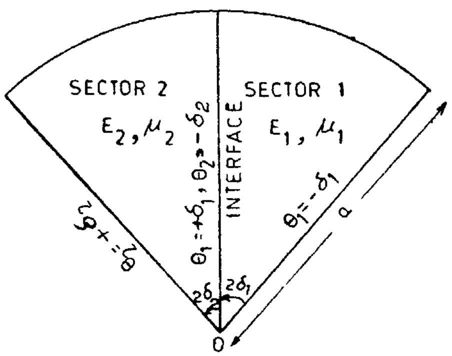

Fig. 2. Bonded sector plate.

denote the quantities pertaining to the plates 1 and 2 by subscripts 1 and 2 respectively. For a perfect bond between the two laminae, the tractions and displacements will be continuous across the interface. Then

$$[\sigma_{\theta_1}, \tau_1]_{\theta_1 = \delta_1} = [\sigma_{\theta_2}, \tau_2]_{\theta_2 = -\delta_2} \qquad \ldots(15a)$$

$$[u_1, v_1]_{\theta_1 = \delta_1} = [u_2, v_2]_{\theta_2 = -\delta_2} \qquad \ldots(15b)$$

From (8) and (15*a*), we get

$$p_1 = p_2 => \frac{\lambda_1}{\delta_1} = \frac{\lambda_2}{\delta_2}; \quad A_2 = B_1,\; C_2 = D_1. \qquad \ldots(16)$$

Equation (16) gives the relationship that must exist between λ_1 and λ_2 in order to ensure the continuity of stresses and displacements across the interface. If the radial edges of the bonded plate are free of tractions (i.e., $A_1 = C_1 = B_2 = D_2 = 0$), then substituting (14) into (15*b*) gives two homogeneous equations for B_1 and D_1.

$$\begin{aligned} 0 = &- D_1 [(\sin 4\lambda_1 - p_1 \sin 4\delta_1)/d_1 + E'(\sin 4\lambda_2 - p_2 \sin 4\delta_2)/d_2] \\ &+ B_1(p_1 + 1) [- \beta_1 + E'\beta_2 + 2(\sin^2 2\lambda_1 - p_1 \sin^2 2\delta_1)/d_1 \\ &- 2E'(\sin^2 2\lambda_2 - p_2 \sin^2 2\delta_2)/d_2] \end{aligned} \quad \ldots(17a)$$

$$\begin{aligned} 0 = &D_1 [- \beta_1 + E'\beta_2 + 2(\sin^2 2\lambda_1 + p_1 \sin^2 2\delta_1)/d_1 \\ &- 2E'(\sin^2 2\lambda_2 + p_2 \sin^2 2\delta_2)/d_2] \\ &+ B_1(p_1 + 1) [(\sin 4\lambda_1 + p_1 \sin 4\delta_1)/d_1 \\ &+ E'(\sin 4\lambda_2 + p_2 \sin 4\delta_2)/d_2]. \end{aligned} \quad \ldots(17b)$$

where $E' = \dfrac{E_1}{E_2} \dfrac{\alpha_2}{\alpha_1}, \; p_2 = p_1, \; \lambda_2 = \dfrac{\lambda_1}{\delta_1} \delta_2 = p_1\delta_2$

$$d_i = \sin^2 2\lambda_i - p_i^2 \sin^2 2\delta_i$$

$$\begin{aligned} \beta_i &= (1 + \mu_i)/\alpha_i \\ &= 1 + \mu_i, \text{ plane stress} \\ &= 1/(1 - \mu_i), \text{ plane strain.} \end{aligned}$$

Non-trivial solutions of (17) exist only for those values of λ_i for which the determinant of the coefficient matrix vanishes. After some simplification, the determinant reduces to

$$\begin{aligned} 0 = &4d_2 + 4E'^2 d_1 + d_1 d_2(\beta_1 - E'\beta_2)^2 \\ &- 4(\beta_1 - E'\beta_2) (d_2 \sin^2 2\lambda_1 - E'd_1 \sin^2 2\lambda_2) \\ &+ 8E' [\sin 2\lambda_1 \sin 2\lambda_2 \cos 2(\lambda_1 + \lambda_2) \\ &-(\lambda_1/\delta_1)^2 \sin 2\delta_1 \sin 2\delta_2 \cos 2(\delta_1 + \delta_2)]. \end{aligned} \quad \ldots(18)$$

This is the required eigenvalue equation on λ_1 and λ_2. Roots of this equation are the values of λ_1 and λ_2 for which a solution of this problem exists in the assumed form. λ_2 can be eliminated from (18) by use of eqn. (16) and an equation in λ_1 is obtained which depends only on the two elastic parameters E' and $(\beta_1 - E'\beta_2)$. By making appropriate changes in the parameters, it can be easily established that eqn. (18) is identical to the one obtained by Bogy (1971) for the two edge-bonded elastic wedges of different materials. It may be noted through eqns. (7) that the stresses will become singular at the apex of the bonded plate as $r \to 0$ whenever $0 < \mathrm{Re}\,(p) < 1$. These singularities have been extensively discussed by Bogy for various wedge angles and combinations of elastic parameters. He has shown that no root p_1, $0 < \text{Re}(p_1) < 1$, of (18) exists for $\delta_1, \delta_2 \leqslant \pi/8$ and only one real root p_1, which depends on elastic parameters, occurs for $\pi/8 \leqslant (\delta_1, \delta_2) \leqslant \pi/4$. For $\pi/4 \leqslant (\delta_1, \delta_2) \leqslant \pi/2$, the root p_1 is real or complex depending on elastic parameters. Furthermore, there may be more than one root p_1 of (18), $0 \leqslant \text{Re}(p_1) \leqslant 1$, for some angles δ_1, δ_2 and material combinations. In fact, the number of such roots may be up to four for some combinations.

The roots of eqn. (18) are symmetrically located in all the four quadrants of the complex plane which means that if λ is a root, so are $-\lambda$, λ^* and $-\lambda^*$, the asterisk (*) denoting the complex conjugate. The roots of this equation are determined for $\mu_1 = \mu_2 = 0.3$, $E_1/E_2 = 2.0$ and $\delta_1 = \delta_2 = \pi/8$ for the state of plane strain and plane stress by using the grid method (see Thompson and Little 1970). The first 15 values of λ_1 for both the cases are given in Table I.

TABLE I

First quadrant eigenvalues of the characteristic equation (18)

$\mu_1 = \mu_2 = 0.3$, $E_1/E_2 = 2.0$, $\delta_1 = \delta_2 = \pi/8$, $\delta_1/\delta_2 = 1.0$, $\lambda_{1n} = \lambda_{2n} = \lambda_n$

	Plane strain		Plane stress	
n	Re λ_n	Im λ_n	Re λ_n	Im λ_n
1	1.132736	0.393131	1.112759	0.399768
2	1.719526	0.000000	1.853032	0.000000
3	2.149540	0.656334	2.137986	0.623233
4	2.985786	0.783162	2.984492	0.741535
5	3.789807	0.969205	3.790257	0.947149
6	4.594939	1.016459	4.596459	0.983119
7	5.387886	1.149975	5.388831	1.130238
8	6.185829	1.169403	6.187423	1.138322
9	6.975409	1.280175	6.976325	1.261304
10	7.769882	1.284797	7.771323	1.254689
11	8.557880	1.382576	8.558706	1.364130
12	9.350296	1.377872	9.351576	1.348278
13	10.137407	1.467175	10.138145	1.448970
14	10.928470	1.456016	10.929610	1.426729
15	11.715033	1.539331	11.715694	1.521277

SATISFACTION OF BOUNDARY CONDITIONS ON THE CIRCULAR EDGE

Once the eigenvalues have been determined for a given set of parameters, it remains finally to compute the eigenfunctions and combine them in such a manner so as to satisfy the boundary conditions on the circular edge $r = a$. It is important to note that each eigenfunction is in fact representing two separate stress functions—one function in plate 1 (with $\lambda = \lambda_1$) and another in plate 2 (with $\lambda = \lambda_2$). We shall imply both functions with the single term "eigenfunction". The stresses in each sector plate are given by (7) with the following values of a_i's from (9) :

Sector Plate 1

$$A_1 = C_1 = 0,\ p = p_1,\ \delta = \delta_1,\ \lambda = \lambda_1$$

$$a_1 = D_1 [(B_1/D_1)(p_1 + 1) \sin (p_1 + 1) \delta_1 - \cos (p_1 + 1) \delta_1]/S_1^+$$

$$a_2 = D_1 [(B_1/D_1)(p_1 + 1) \cos (p_1 + 1) \delta_1 + \sin (p_1 + 1) \delta_1]/S_1^-$$

$$a_3 = - D_1 [(B_1/D_1)(p_1 - 1) \sin (p_1 - 1) \delta_1 - \cos (p_1 - 1) \delta_1]/S_1^+$$

$$a_4 = - D_1 [(B_1/D_1)(p_1 - 1) \cos (p_1 - 1) \delta_1 + \sin (p_1 - 1) \delta_1]/S_1^- \quad ...(19a)$$

Sector Plate 2

$$B_2 = D_2 = 0,\ A_2 = B_1,\ C_2 = D_1,\ p = p_2,\ \delta = \delta_2,\ \lambda = \lambda_2$$

$$a_1 = D_1 [(B_1/D_1)(p_2 + 1) \sin (p_2 + 1) \delta_2 + \cos (p_2 + 1) \delta_2]/S_2^+$$

$$a_2 = - D_1 [(B_1/D_1)(p_2 + 1) \cos (p_2 + 1) \delta_2 - \sin (p_2 + 1) \delta_2]/S_2^-$$

$$a_3 = - D_1 [(B_1/D_1)(p_2 - 1) \sin (p_2 - 1) \delta_2 + \cos (p_2 - 1) \delta_2]/S_2^+$$

$$a_4 = D_1 [(B_1/D_1)(p_2 - 1) \cos (p_2 - 1) \delta_2 - \sin (p_2 - 1) \delta_2]/S_2^- \quad ...(19b)$$

The ratio B_1/D_1 can be obtained from either of the eqns. (17*a, b*) and it depends only on the eigenvalue pair (λ_1, λ_2), the angles (δ_1, δ_2) and the elastic moduli.

$$B_1/D_1 = [d_2(\sin 4\lambda_1 - p_1 \sin 4\delta_1) + E'd_1(\sin 4\lambda_2 - p_2 \sin 4\delta_2)]/[(p_1 + 1) \times \{d_1 d_2(- \beta_1 + E'\beta_2) + 2d_2(\sin^2 2\lambda_1 - p_1 \sin^2 2\delta_1) - 2E'd_1(\sin^2 2\lambda_2 - p_2 \sin^2 2\delta_2)\}]. \quad ...(20)$$

Each eigenfunction is a solution of the differential eqn. (1) for the stresses in bonded sector plate when its radial edges are stress-free. Since the equations are linear, a linear combination of all the eigenfunctions is also a solution. In order that the stresses remain bounded for all values of r, $0 \leqslant r/a \leqslant 1$, the summation is restricted over the pair of eigenvalues lying in the right half of the complex plane, i.e., whose real part is positive.

For any arbitrary loading on the circular boundary $r = a$, the general solution of $\phi(r, \theta)$ will be sought in the form

$$\phi(r, \theta) = \sum_{n=1}^{\infty} D_{1n}(r/a)^{(p_n+1)} F_n(\theta), \text{ Re }(p_n) > 0. \qquad \ldots(21)$$

Using the constants a_i's from (19) and the relations (7*a*, *c*), the most general form of the boundary stresses on $r = a$ can be expressed in the form

$$\sigma_{rb} = \sum_{n=1}^{\infty} D_{1n}\sigma_{rn} \qquad \ldots(22a)$$

$$\tau_b = \sum_{n=1}^{\infty} D_{1n}\tau_n \qquad \ldots(22b)$$

where

$$\sigma_{rn} = (p_n + 1) F_n(\theta) + F''_n(\theta) \qquad \ldots(23a)$$

$$\tau_n = -p_n F'_n(\theta) \qquad \ldots(23b)$$

the suffix b denotes the prescribed boundary function. We remember that in the sector plate 1, $\lambda = \lambda_{1n}$ and the constants a_i's are those belonging to this plate (i.e., given by (19*a*)) and in sector plate 2, $\lambda = \lambda_{2n}$ and the constants a_i's are given by (19*b*).

For a prescribed self-equilibrating load on the circular boundary $r = a$, the constants D_{1n} are determined using the method of least squares (see Thompson and Little 1972). For illustrating the numerical results, the following example has been considered.

Numerical Results

We assume that the elastic parameters are given by $\mu_1 = \mu_2 = 0.3$ and $E_1/E_2 = 2.0$. We set $\delta_1 = \delta_2 = \delta = \pi/8$ so that $\delta_1/\delta_2 = 1$ and hence $\lambda_{1n} = \lambda_{2n}$.

Consider the following self-equilibrating load on $r = a$, viz. :

For Sector Plate 1

$$\sigma_{rb} = \frac{\pi}{8\delta} \sin\left(\frac{\pi\theta}{2\delta}\right), \quad \tau_b = -\tfrac{1}{4}\cos\left(\frac{\pi\theta}{2\delta}\right) \qquad \ldots(24a)$$

For Sector Plate 2

$$\sigma_{rb} = -\frac{\pi}{8\delta} \sin\left(\frac{\pi\theta}{2\delta}\right), \quad \tau_b = \tfrac{1}{4}\cos\left(\frac{\pi\theta}{2\delta}\right). \qquad \ldots(24b)$$

The constants D_{1n} are determined, using the first five, ten and fifteen pairs of eigenvalues for the state of plane strain. The numerical convergence of normal and shear stresses is presented in Table II. As expected, the fit is improved by increasing

TABLE II

Numerical convergence of the eigenfunction expansion for the prescribed load (24*a*, *b*) *on* $r = a$

θ_i/δ_i	Applied normal stress σ_{rb}	Reproduced normal stress			Applied shear stress b	Reproduced shear stress		
		5 Eigen-values	10 Eigen-values	15 Eigen-values		5 Eigen-values	10 Eigen-values	15 Eigen-values
Plate 1								
− 1.0	− 1.0000	− 1.0810	− 1.0087	− 0.9757	0.0000	0.0000	0.0000	0.0000
− 0.6	− 0.8090	− 0.8057	− 0.7963	− 0.8068	− 0.1469	− 0.1870	− 0.1389	− 0.1501
− 0.2	− 0.3090	− 0.2756	− 0.3134	− 0.3088	− 0.2378	− 0.1842	− 0.2386	− 0.2423
0.0	0.0000	0.0330	− 0.0079	− 0.0009	− 0.2500	− 0.2292	− 0.2627	− 0.2489
0.2	0.3090	0.3137	0.3101	0.3158	− 0.2378	− 0.2575	− 0.2370	− 0.2425
0.6	0.8090	0.7841	0.8102	0.8103	− 0.1469	− 0.1650	− 0.1673	− 0.1487
1.0	1.0000	1.1346	1.0935	1.0822*	0.0000	0.0917	0.0632	0.0485
Plate 2								
− 1.0	1.0000	1.0003	0.9719	0.9676*	0.0000	0.0917	0.0632	0.0485
− 0.6	0.8090	0.7655	0.8011	0.8179	0.1469	0.1212	0.1355	0.1377
− 0.2	0.3090	0.3044	0.3214	0.3103	0.2378	0.2527	0.2420	0.2362
0.0	0.0000	0.0410	− 0.0041	0.0004	0.2500	0.2285	0.2599	0.2473
0.2	− 0.3090	− 0.2656	− 0.3194	− 0.3048	0.2378	0.1761	0.2393	0.2431
0.6	− 0.8090	− 0.8041	− 0.7944	− 0.8029	0.1469	0.1871	0.1360	0.1456
1.0	− 1.0000	− 1.0923	− 1.0025	− 0.9834	0.0000	0.0000	0.0000	0.0000

*It may be noted that the reproduced normal stress σ_r is discontinuous on the interface while the applied stress is continuous.

the number of eigenvalues, i.e., by adding more number of eigenfunctions in the series expansion. It is important to note that the reproduced σ_r is discontinuous along the interface, while the applied σ_r is continuous. This is a direct consequence of the fact that the eigenfunctions result in the stress distribution $\sigma_r(\theta)$ that are discontinuous across the interface (except when the two materials are identical).

The interface normal stress σ_θ is illustrated graphically in Fig. 3. As can be observed through the Fig. 3, the normal stress σ_θ changes from tension to compression at a distance equal to 3/10th of the radius r/a from the loaded end of the sector plate and then decays out to zero.

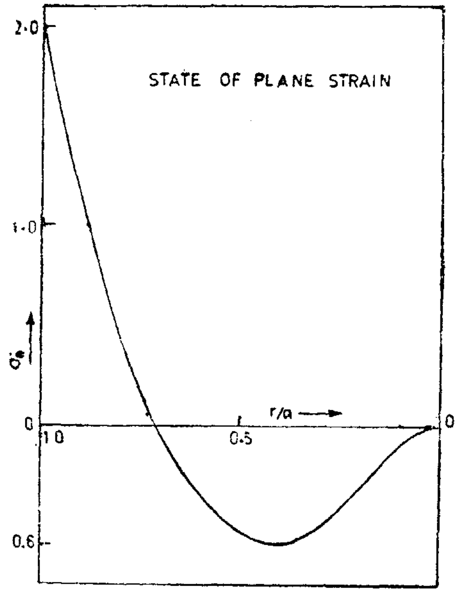

FIG. 3. Normal stress σ_θ on the interface of the bonded sector plate for the applied traction (24).

The numerical results depend upon the various parameters such as the ratio E_1/E_2, the Poisson's ratios μ_1, μ_2 and the sector angles α_1 and α_2. On the basis of the numerical results obained for some values of the parameters, we conclude that the variations in the Poisson's ratios do not make any significant change in the interface stresses excepting in the neighbourhood of the interface corners $r = a$, $\theta = 0$, where there is a distinct jump. This may be attributed to the stress singularity which exists at the corner of the dissimilar materials. On account of this singular behaviour, the numerial results are doubtful at the interface corners. The interface stresses are sensitive to changes in the angles α_1, α_2 and the ratio E_1/E_2.

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