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LOCAL AND GLOBAL BEHAVIOUR OF ACCELERATION WAVES IN RADIATING GASES

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This paper deals with the propagation of acceleration waves in gas flows with the radiant heat flux. A differential equation governing the local and global behaviour of the amplitude of the wave is derived and solved. The dependence of wave amplitude on time is studied and its local and global behaviour with respect to growth and decay of the wave is discussed. The problem of breakdown of the wave is studied. It is found that there exists a critical value of the initial wave amplitude such that all compressive waves with an initial amplitude greater than the critical one with breakdown after a finite critical time t_c and a shock type discontinuity will be formed. The point of breakdown occurs at the cusp of the envelope of intersecting forward characteristics. On the other hand all expansion waves and compressive waves with initial amplitude less than the critical one will decay out.

1. INTRODUCTION

One of the most interesting problems of the theory of propagation of an acceleration wave in a continuous media is the process of formation of shock waves, which has been extensively studied during the last decade. Becker (1970), Bowon and Chen (1972) and Ram and Gaur (1976) studied the various properties of the acceleration wave in non-equilibrium flows. Sahubi and Jeffrey (1976), Varley (1965) and Green (1964) studied the growth and decay of acceleration waves in different varieties of materials. Coleman and Gurtin (1965) studied their local and global behaviour of the same in materials with fading memory. Rarity (1967) used Jeffrey's technique to study the problem of breakdown of characteristic solutions in flows with vibrational relaxation. The main academic interest of the present communication is to use Jeffrey's technique for investigating the effects of radiation under optically thin gas approximation (Olfc 1969) on the global behaviour of an acceleration wave.

2. DEPENDENCE OF WAVE AMPLITUDE ON TIME

Let us consider a symmetric gas flow with radiative heat flux effects, which is induced by the motion of a piston advancing with finite acceleration into a constant state at rest. The basic equations governing the gas flow are

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} + \delta\frac{\rho u}{x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} + (\gamma - 1)\,\alpha_p a_R T^4 = 0$$

where u, p, ρ, T represent the translational properties of the gas and have their usual meanings. α_p is the Planck mean absorption coefficient. For a gray gas the absorption coefficient does not vary with frequency and can be treated as a constant. x is the radial distance from the origin of symmetry and $\delta = 0$, 1 and 2 for planar, cylindrical and spherical symmetry respectively.

Now combining these basic equations we have

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B} = 0 \qquad \ldots(1)$$

where **U** and **B** are the column vectors and **A** is a matrix of order three which are given by

$$\mathbf{U} = \begin{bmatrix} p \\ u \\ \rho \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & \gamma p & 0 \\ \frac{1}{\rho} & u & 0 \\ 0 & \rho & u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} (\gamma - 1)\,\alpha_p a_R T^4 \\ 0 \\ \delta\rho u/x \end{bmatrix}$$

The system (1) is quasilinear and has three real characteristics along which weak discontinuities, that is acceleration waves, propagate. The eigen values of the matrix A

$$\lambda^1 = u + C, \; \lambda^2 = u - C, \; \lambda^3 = u$$

and the corresponding eigen vectors are given by

$$\mathbf{L}^1 = \begin{bmatrix} 1 \\ C\rho \\ C^2 \end{bmatrix}, \quad \mathbf{L}^2 = \begin{bmatrix} 1 \\ -C\rho \\ C^2 \end{bmatrix}, \quad \mathbf{L}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now we introduce curvilinear coordinates (ϕ, t') by the equations

$$\phi_t + \lambda^1\phi_x = 0, \; t = t'. \qquad \ldots(2)$$

In view of (2) the leading forward characteristic front can be represented by

$$\phi(x, t) = 0$$

if we level the family of characteristics by $\phi = t^*$, where t^* is the time when a wave front is produced by the piston. Any flow property $f(x, t)$ is continuous across $\phi(x, t) = 0$, but $\partial f/\partial x$ and $\partial f/\partial t$ undergo finite jumps across it. Such discontinuities are defined as "acceleration waves" advancing normal to themselves with the speed of sound relative to the gas flow. The transformation introduced through eqn. (2) is non-singular, provided the Jacobian of the transformation, i.e.

$$J = x_\phi = 1/\phi_x \qquad \ldots(3)$$

is non-zero and finite. Since x_ϕ is initially equal to unity, therefore we can expect that the transformation is valid at least for a finite time. Since J vanishes when $x_\phi = 0$, we expect an acceleration wave to breakdown at the point of intersection of forward moving characteristics. Let us consider an open region R bounded by two characteristics $\phi(x, t) = 0$ and $\xi(x, t) = 0$ such that no characteristics issuing from the origin enters this open region R. We assume that $\mathbf{U}$ remains smooth in the region R atleast for a finite time during which the transformation is non-singular (Fig. 1).

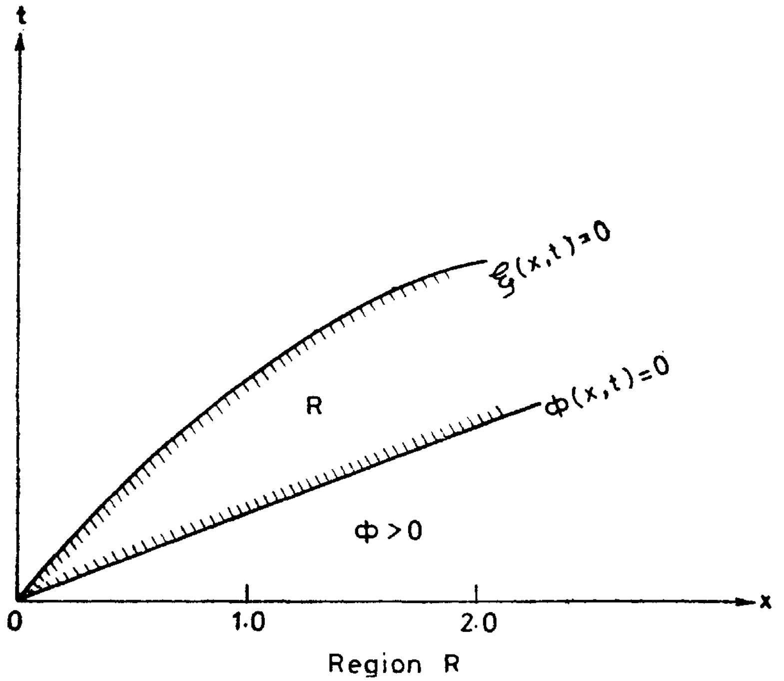

FIG. 1. Region of influence.

Transforming (1) into the new coordinate system (ϕ, t') and pre-multiplying by $\mathbf{L}^j$, such that $\mathbf{b}^j = \mathbf{L}^j\,\mathbf{B}$, we have

$$\mathbf{L}^j \left\{ x_\phi \frac{\partial}{\partial t'} + (\lambda - \lambda^1) \frac{\partial}{\partial \phi} \right\} \mathbf{U} + x_\phi\, \mathbf{b}^j = 0 \qquad \ldots(4)$$

which provides

$$\mathbf{L}^1\, \mathbf{U}_{t'} + \mathbf{b}^1 = 0. \qquad \ldots(5)$$

The boundary conditions at the wave front $\phi(x, t) = 0$ are

$$[\mathbf{U}]_{\phi=0+}^{\phi=0-} = 0,\ [\mathbf{U}_{t'}]_{\phi=0+}^{\phi=0-} = 0,\ [\mathbf{U}_{\phi}]_{\phi=0+}^{\phi=0-} \neq 0 = \underset{\sim}{\pi}(t'),\ [x_{\phi}]_{\phi=0+}^{\phi=0-} \neq 0 = X(t')$$

where

$$[Z]_{\phi=0+}^{\phi=0-} = Z(0^-, t') - Z(0^+, t')$$

From the definition of $X(t') \neq 0$ we see at once that

$$X(t') + (x_{\phi})_{\phi=0+} = (x_{\phi})_{\phi=0-} \qquad \text{...(6)}$$

while $(x_{\phi})_{\phi=0+} = (x_{\phi})_0$ is finite. Therefore, the condition (3) implies that the right-hand side of (6) is finite and non-zero.

Since $\mathbf{U}_0$ corresponds to the constant state condition ahead of the wave front we have $\mathbf{B}(\mathbf{U}_0) = 0$. It follows immediately from (4) that

$$\mathbf{L}^j \underset{\sim}{\pi}(t') = 0, (\lambda \neq \lambda') \qquad \text{...(7)}$$

Differentiating eqn. (5) with respect to ϕ at any point of the open region R and allowing it to tend to a point on the wave front $\phi(x, t) = 0$, we get

$$\mathbf{L}_0^1 \underset{\sim}{\pi}_{t'}(t') + [\nabla_u(\mathbf{L}^1\mathbf{B})]_0 \underset{\sim}{\pi} = 0 \qquad \text{...(8)}$$

where ∇_u denotes the gradient operator with respect to the components of the vector $\mathbf{U}$.

The equation of the outgoing characteristics can be written as

$$\partial x/\partial t' = \lambda^1 = u + C. \qquad \text{...(9)}$$

Differentiating this result with respect to ϕ at any point in the open region R we obtain

$$\partial(x_{\phi})/\partial t' = (\nabla_u \lambda^1)\, \mathbf{U}_{\phi}$$

and allowing this point to tend to a point on the wave front and using the boundary conditions, we get

$$X_{t'} = [\nabla_u(\lambda^1)]_0 \underset{\sim}{\pi}$$

which on integration provides that

$$X = \bar{X} + \int_0^t [\nabla_u(\lambda^1)]_0 \underset{\sim}{\pi}\, dt' \qquad \text{...(10)}$$

where

$$\bar{X} = \lim_{t' \to 0} X.$$

We would expect the solution to breakdown after a finite critical time t_c determined by the intersection of two outgoing characteristics emanating from the piston. At such a critical point the Jacobian of transformation must vanish by virtue of (3) and therefore, we have

$$X(t') + (x_\phi)_0 = 0$$

so that t_c is given by

$$1 + \int_0^{t_c} [\nabla_u(\lambda^1)]_0 \, \underset{\sim}{\pi}/\bar{x}_\phi \, dt' = 0 \qquad \text{...(11)}$$

In consequence of (7) we have

$$\pi_1 = C_0\rho_0\pi_2, \; \pi_3 = 0. \qquad \text{...(12)}$$

Also we have

$$[\nabla_u(\mathbf{L}^1\mathbf{B})]_0 \, \underset{\sim}{\pi} = \frac{4(\gamma - 1)\, \alpha_p a_R T_0^4}{p_0} \pi_1 + \frac{C_0^2 \delta \rho_0 K(0)}{(1 + C_0 K(0) t)} \pi_2. \qquad \text{...(13)}$$

Using (12) and (13) in (8) we get

$$\pi_{2t'} + C_1 \pi_2 + \frac{K(0)\, C_0 C_2}{(1 + C_0 K(0)\, t)} \pi_2 = 0 \qquad \text{...(14)}$$

where

$$C_1 = \frac{2(\gamma - 1)\, \alpha_p a_R T_0^4}{p_0} \text{ and } C_2 = \delta/2$$

are constants of the constant state $\mathbf{U}_0$. $K(0)$ is the curvature of the initial wave front. Thus the jump discontinuities $(\mathbf{U}_\phi]_{\phi=0+}^{\phi=0-} = \underset{\sim}{\pi}$ across $\phi = 0$ are dependent of time and can be expressed as

$$\underset{\sim}{\pi} = \bar{\pi}_2 \exp\,(-C_1 t)\,(1 + C_0 K(0)\, t)^{-C_2} \begin{bmatrix} C_0\rho_0 \\ 1 \\ 0 \end{bmatrix}. \qquad \text{...(15)}$$

Substituting from (15) in (11) we get

$$1 + \frac{(\gamma + 1)}{2} \, \bar{u}_x \int_0^{t_c} \exp\,(-C_1 t)\,(1 + C_0 K(0)\, t)^{-C_2} \, dt = 0$$

which determines the critical time t_c for an acceleration wave to breakdown and consequently for a shock wave to appear.

3. Growth Equation

Differentiating (9) with respect to ϕ at any point in R and allowing this point to tend to the wave front $\phi(x, t) = 0$ we get

$$x_{\phi t'}/x_\phi = [\nabla_u(\lambda^1)]_0 \underset{\sim}{\pi}/x_\phi = \tfrac{1}{2}(\gamma + 1)\, \pi_2/x_\phi. \quad ...(16)$$

The amplitude $a(t)$ of an acceleration wave is defined by

$$a(t) = [u_x]_{\phi=0+}^{\phi=0-} = \pi_2/x_\phi. \quad ...(17)$$

The equation (14) provides that

$$u_{\phi t'}/x_\phi = -\{C_1 + C_0 K(0)\, t/(1 + C_0 K(0) t)\}\, a(t). \quad ...(18)$$

Differentiating (17) with respect to t and making use of (16) and (18) we get

$$\frac{da(t)}{dt} + \{C_1 + C_0 K(0)\, C_2/(1 + C_0 K(0) t)\}\, a(t) + \frac{(\gamma + 1)}{2} a^2(t) = 0 \quad ...(19)$$

which is the fundamental growth equation governing the local and global behaviour of an acceleration wave.

The growth eqn. (19) can be written for $\gamma = 1.4$ in the dimensionless form as

$$\frac{d\eta}{d\tau} + \{C_1 + C_2/(1 + \tau)\}\, \eta + 1.2\, C_3 \eta^2 = 0 \quad ...(20)$$

where $\eta = a(t)/a(0)$, $\tau = C_0 K(0)\, t$ and $C_3 = a(0)/C_0 K(0)$ are non-dimensional parameters. The integral curves of (20) are shown in Figs. 2, 3 and 4 for different cases. Integral curves in Fig. 2 show that the finite increase in the piston acceleration amounts to an early shock formation. The solution curves of Fig. 3 show that the effects of radiation heat transfer cause delay in the shock formation and thus the point of breakdown moves outwards along the leading characteristics. The integral curves in Fig. 4 show that curvature effects of the wave-geometry are to increase the critical value of $a(0)$ above which there occurs no breakdown.

4. Local and Global Behaviour of Amplitude

Let us assume that the distance travelled by the wave during the interval of time t is σ so that $\sigma = tC_0$. Thus the solution of the growth eqn. (19) can be written as

$$a(\sigma) = \frac{\exp(-C_1 \sigma/C_0)\,(1 + \sigma K(0))^{-C_2}}{\dfrac{1}{a(0)} + \dfrac{(\gamma + 1)}{2C_0} \displaystyle\int_0^\sigma \exp(-C_1 x/C_0)\,(1 + xK(0))^{-C_2}\, dx}. \quad ...(21)$$

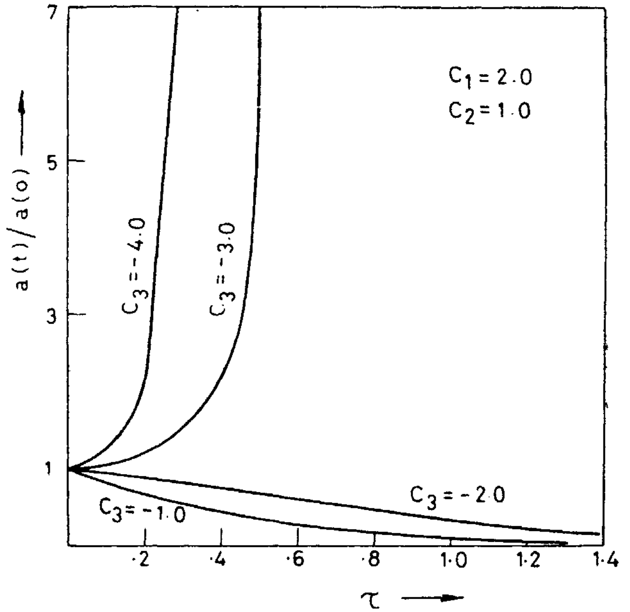

FIG. 2. Effect of the piston acceleration on the growth of the acceleration waves.

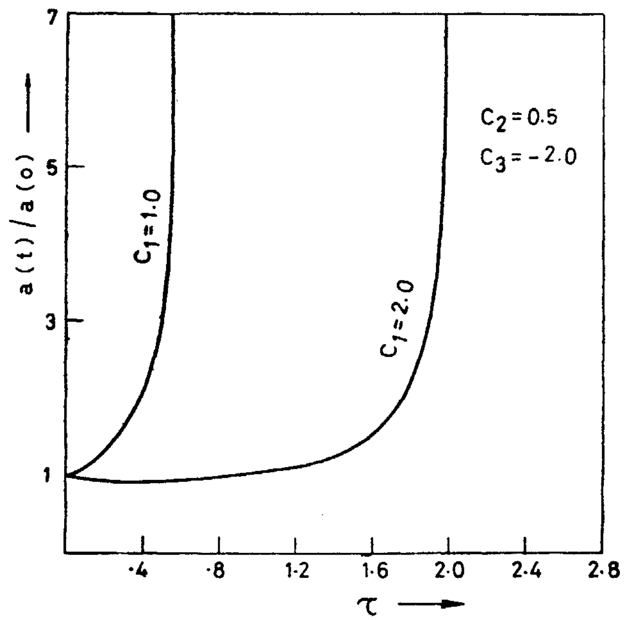

FIG. 3. Radiation effect on the growth of the acceleration waves.

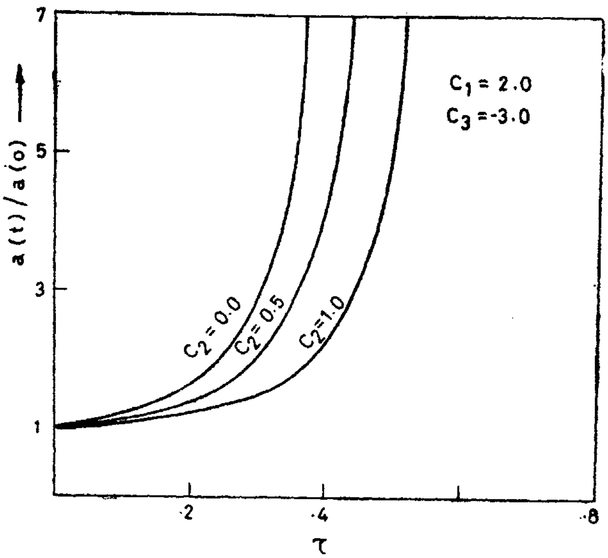

FIG. 4. Effect of the wave geometry on the growth of the acceleration waves.

In order to study the local and global behaviour of $a(\sigma)$ we neglect higher powers of σ, so that (21) can be put into the form

$$a(\sigma) = 1 - \left(\frac{C_1}{C_0} + C_2 K(0) + \frac{(\gamma + 1)\, a(0)}{2C_0}\right) \sigma + O(\sigma^2). \qquad \ldots(22)$$

The relation (22) shows if $a(0) > 0$, the amplitude will decrease, but if $a(0) < 0$ and $|a(0)| > a_c$, such that

$$a_c = \frac{2C_0}{(\gamma + 1)} \left(\frac{C_1}{C_0} + C_2 K(0)\right)$$

then $a(\sigma)$ will grow. If $a(0) < 0$ and $|a(0)| < a_c$, then $a(\sigma)$ will again decrease.

For the global behaviour of the amplitude $a(\sigma)$ we observe that $a(\sigma)$ will continuously decay for expansion waves ($a(0) > 0$). A compressive wave with $a(0) < 0$ will, in general, continuously grow and ultimately terminate into a shock wave. In this case there exists a critical value of $a(0)$ given by

$$a_c(0) = \left\{ \frac{(\gamma + 1)}{2C_0} \int_0^\infty \exp\left(- C_1 \sigma / C_0\right) (1 + \sigma K(0))^{-C_2}\, d\sigma \right\}^{-1} \qquad \ldots(23)$$

such that the formation of the shock wave will take place for $|a(0)| > a_c(0)$, but there occurs no shock formation for $|a(0)| < a_c(0)$.

5. Special Case of Interest

It is evident from the growth equation that the contribution of the wave geometry is only in one term which also contains the contribution of radiation. Hence the curvature effects will modify the radiation effects to some extent. Therefore, we shall now confine our analysis to a special case in order to study the global behaviour of the wave in details. Now we consider a plane wave case for which $C_2 = 0$ and the growth equation reduces to the form

$$\frac{da(t)}{dt} + C_1 a(t) + \frac{(\gamma + 1)}{2} a^2(t) = 0 \qquad \dots(24)$$

The solution of (24) is of the form

$$a(t) = a(0)\{\exp(C_1 t) \,.\, (1 + a(0)/a_c) - a(0)/a_c\}^{-1} \qquad \dots(25)$$

where

$$a_c = 2C_1/(\gamma + 1)$$

is the critical value of $a(0)$. If an acceleration wave is a compressive wave ($a(0) < 0$) and $|\, a(0) \,| < a_c$, $a(t)$ decreases exponentially and tends to zero as $t \to \infty$. If $a(0) = a_c$, $a(t)$ becomes independent of time and is constant for all times. If $|\, a(0) \,| > a_c$, $a(t)$ increases exponentially and tends to infinity within a finite time t_c given by

$$t_c = \frac{1}{C_1} \log \left\{\frac{a(0)}{a(0) - a_c}\right\}. \qquad \dots(26)$$

Thus when $a(0)$ exceeds a_c, then there occurs a breakdown of the acceleration wave and consequently a shock wave will be formed. In case of an expansion wave ($a(0) > 0$) there occurs no breakdown and the wave will decay and will be damped out ultimately.

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