Indian J. pure appl. Math., **10**(8) : 959-963, August 1979

ON THE PROPAGATION OF A CYLINDRICAL MHD BLAST WAVE

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(*Received* 13 *November* 1978)

The propagation of a cylindrical MHD shock wave, during the stage when the wave is still strong (in the hydrodynamical sense) has been studied. The variations of the flow and field parameters at the rear of the wave are determined in terms of its radius R and the time derivatives of R. The dependence of the speed of propagation on the explosive energy, the radius of the wave and the magnetic field is determined and some interesting inferences are drawn. The law of propagation obtained by Lin (1954) has been recovered in the limit of vanishing magnetic field.

1. INTRODUCTION

Taylor (1950), Sakurai (1953), Thomas (1957a), Sedov (1959) and others investigated the problem of an explosion wave in air. The results obtained by these authors have proved to be very useful in many other fields, especially in the application of the theory to the hypersonic flow problems and in describing very successfully the strong blast wave phenomenon resulting from an atomic explosion. Carrus *et al.* (1951) and Kopal (1954) accounted for the effects of gravitation and investigated by a purely numerical method, similarity solutions in which the flow was headed by a shock wave. Lin (1954) extended the Taylor's analysis to a cylindrical case and showed that his analysis is not only applicable to the one-dimensional situation but also to certain axially symmetric hypersonic flow problems, such as the shock envelope behind a fast meteor or missible. In the present paper, an attempt has been made to determine the law of propagation of a very strong (in the hydrodynamical sense) cylindrical shock wave under the influence of an axial magnetic field. In the limit of zero magnetic field, the law of propagation obtained by Lin (1954) has been recovered. It is assumed that a cylindrical shock wave is produced by the sudden release of a finite amount of energy per unit length along a straight line of infinite extent in a perfectly conducting gas. An axial magnetic field is assumed to exist initially in the conducting gas. As a consequence of the assumption of infinite conductivity, the magnetic field in the gaseous flow remains axial. Further, it is assumed that the gas is devoid of viscosity and thermal conductivity.

2. Basic Equations, Shock Relations and Compatibility Conditions

The equations governing the flow are

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial r} + u\left(\frac{\partial \rho}{\partial r} + \frac{\rho}{r}\right) = 0 \qquad \text{...(1)}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial r} + \frac{\partial p^*}{\partial r} = 0 \qquad \text{...(2)}$$

$$\frac{\partial H}{\partial t} + u \frac{\partial H}{\partial r} + H\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0 \qquad \text{...(3)}$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + \gamma p\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0 \qquad \text{...(4)}$$

where $p^* = p + (\mu H^2/2)$ and the variables H, u, ρ, p are respectively the axial magnetic field, the radial velocity, the density and the gas pressure in the flow at the radial distance r and the time t.

The motion is bounded on the outside by the shock surface at $r = R(t)$ which propagates radially outwards into a quiescent gas with a radial velocity $G = dR/dt$. We shall denote by H_0, p_0, ρ_0 and u_0 $(= 0)$ the constant values of the field, pressure, density and the gas velocity respectively ahead of the shock and by H, p, ρ, u their respective values immediately behind the moving shock front. When the shock wave remains sufficiently strong, the pressure, density, velocity and field immediately behind the shock are determined in terms of their upstream values by the following strong shock conditions

$$[u] = \frac{2G}{\gamma + 1},\ [p^*] = \frac{2\rho_0 G^2}{\gamma + 1},\ \ [\rho] = \frac{2\rho_0}{\gamma - 1},\ [H] = \frac{2H_0}{\gamma - 1} \qquad \text{...(5)}$$

where the square bracket enclosing a quantity stands for the value of the quantity immediately behind the shock surface minus its value just ahead of the shock surface.

Compatibility conditions of first order for a singular surface have been derived by Thomas (1957b), which in the present case reduce to the following form :

$$\left[\frac{\partial f}{\partial t}\right] = -AG + \frac{\delta B}{\delta t} \qquad \text{...(6)}$$

where

$$A = \left[\frac{\partial f}{\partial r}\right],\ \ B = [f] \qquad \text{...(7)}$$

and the quantity f may represent any of the variables H, p, u and ρ. The δ-time derivative of any quantity f defined over $r = R(t)$, is given by

$$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} + G \frac{\partial f}{\partial r} \qquad \text{...(8)}$$

which suggests that the δ-time derivative of any quantity which is considered to be expressed on $r = R(t)$ as a function of the time alone is identical with the ordinary time derivative of the quantity. For the sake of simplicity, let the jumps in the flow parameters H, p, u and ρ be denoted by k, l, m and n respectively. The δ-time derivatives of k, l, m, and n are given by

$$\frac{\delta k}{\delta t} = \frac{2}{\gamma+1}\frac{dG}{dt};\ \frac{\delta l}{\delta t} = 0;\ \frac{\delta m}{\delta t} = \frac{4\rho_0 G}{\gamma+1}\frac{dG}{dt},\ \frac{\delta n}{\delta t} = 0. \qquad \text{...(9)}$$

3. Flow Variations Behind the Shock

Evaluating eqns. (1) - (4) on the inner boundary of $r = R(t)$ and using (6), (7) and (9), we get

$$(G-u)\,\tau - \rho\lambda = \rho u/R \qquad \text{...(10)}$$

$$\rho(G-u)\,\lambda - \mu H\eta = \xi + \frac{2\rho}{\gamma+1}\frac{dG}{dt} \qquad \text{...(11)}$$

$$(G-u)\,\eta - H\lambda = uH/R \qquad \text{...(12)}$$

$$(G-u)\,\xi - \gamma p\lambda = \frac{\gamma p u}{R} + \frac{4\rho_0 G}{\gamma+1}\frac{dG}{dt} \qquad \text{...(13)}$$

where λ, ξ, τ and η are respectively the values of the derivatives of u, p, ρ and H with respect to the distance r just at the rear of the wave surface. Solving eqns. (10) - (13) for λ, ξ, τ and η, and making use of the relations

$$\rho(G-u) = \rho_0 G,\ H(G-u) = H_0 G \qquad \text{...(14)}$$

we get

$$\lambda = \rho^2\left(\frac{6(\gamma-1)}{(\gamma+1)^2}G\frac{dG}{dt} + \frac{uC_{eff}^2}{R}\right)\Big/(\rho_0^2G^2 - \rho^2C_{eff}^2) \qquad \text{...(15)}$$

$$\xi = c^2\left\{\frac{2}{\gamma+1}\left(\frac{\rho^2c^2 + 2\rho_0^2G^2 - 2\rho^2b^2}{\rho c^2}\right)\frac{dG}{dt} + \frac{\rho_0 uG}{R}\right\}\Big/(\rho_0^2G^2 - \rho^2C_{eff}^2) \qquad \text{...(16)}$$

$$\tau = \rho^2\left(\frac{6\rho}{\gamma+1}\frac{dG}{dt} + \frac{\rho_0 uG}{R}\right)\Big/(\rho_0^2G^2 - \rho^2C_{eff}^2) \qquad \text{...(17)}$$

$$\eta = H_0\rho^2\left(\frac{6}{\gamma+1}\frac{dG}{dt} + \frac{uG}{R}\right)\Big/(\rho_0^2G^2 - \rho^2C_{eff}^2) \qquad \text{...(18)}$$

where $C_{eff}^2 = c^2 + b^2$; c and b are respectively the sound speed $(\gamma p/\rho)^{1/2}$ and the Alfvén speed $(\mu H^2/\rho)^{1/2}$ at the inner boundary of the wave surface.

Substituting the values of u, p, ρ and H in terms of their upstream values with the help of the shock conditions, eqns. (15) - (18) determine the variations of the flow and field parameters with respect to the distance r at the rear of the wave surface.

Similarly, using the above expressions (15) - (18) and the compatibility condition (6), we obtain the formulae for the time derivatives of u, p, ρ and H behind the shock. Thus, we see that the derivatives of the flow and field parameters are given by linear expressions in dG/dt with coefficients as functions of the wave velocity G and the radius R.

4. Wave Velocity

As the speed of propagation G cannot be determined from the shock conditions, an additional assumption is required for its determination. We consider a shock wave to be the result of the blast wave originating from an explosion. Let us assume that the shock surface encloses a volume V and $V + \delta V$ at times t and $t + \delta t$ respectively. Then, the energy in the shell is obviously $\rho E \delta V$, where E is the total energy per unit mass given by

$$E = \frac{p}{\rho(\gamma - 1)} + \tfrac{1}{2} u^2 + \tfrac{1}{2} \mu \frac{H^2}{\rho}. \qquad \text{...(19)}$$

The energy $\rho E \delta V$ in the shell consists of the energy $E_0 \rho \delta V$ derived from the undisturbed gas entering into the shell plus the energy δQ derived from the energy released by explosion. We shall make the assumption that the energy δQ is directly proportional to the total energy Q released by the explosion and δV but inversely proportional to V. Thus, we have

$$\rho E \delta V = \rho E_0 \delta V + \frac{\alpha Q \delta V}{V} \qquad \text{...(20)}$$

where α is the constant of proportionality. In the limit $\delta V \to 0$, eqn. (20) yields

$$[E] = \frac{\alpha Q}{\rho V}. \qquad \text{...(21)}$$

The arguments remain valid for plane, cylindrical and spherical shocks. For these cases, the energy Q is to be interpreted as the energy released per unit area of the plane shock, per unit length of the cylindrical shock and per unit volume of the spherical shock, respectively.

To determine the constant of proportionality, we note that

$$\alpha = \frac{\rho}{\rho_0} \left(\frac{\rho_0 E V}{Q} \right) - \frac{\rho E_0 V}{Q}. \qquad \text{...(22)}$$

In the limit as V approaches zero, eqn. (22) yields

$$\alpha = \operatorname*{Lt}_{V \to 0} \frac{\rho}{\rho_0} = \frac{\gamma + 1}{\gamma - 1}. \qquad \text{...(23)}$$

Since $\operatorname*{Lt}_{V \to 0} (\rho_0 E V) = Q$ and $\operatorname*{Lt}_{V \to 0} (\rho E_0 V) = 0$.

Thus, for a cylindrical blast, eqn. (21) together with (19), (23) and the shock conditions (5) yields

$$G^2 = \frac{(\gamma + 1)^2 Q}{4\pi\rho_0 R^2} + \frac{(\gamma + 1)(1 + 2\gamma - \gamma^2)}{4(\gamma - 1)^2} b_0^2 \qquad \text{...(24)}$$

which determines the velocity of propagation of the cylindrical shock wave during the stage when the wave is still strong. This relation shows that if the value of the heat exponent $\gamma < 1 + \sqrt{2}$, then the magnetic field effects are to increase the velocity of propagation relative to what it would be in the absence of magnetic field. The relation (24) can be transformed into a differential equation by putting $G = dR/dt$, which can be integrated to determine the radius $R(t)$ as a function of the time t in the following form:

$$R = (At + Bt^2)^{1/2} \qquad \text{...(25)}$$

where

$$A = (\gamma + 1)\sqrt{(Q/\pi\rho_0)} \quad \text{and} \quad B = \frac{(\gamma + 1)(1 + 2\gamma - \gamma^2)}{4(\gamma - 1)^2} b_0^2.$$

In the limit of zero magnetic field, eqn. (25) shows that the radius R of a strong cylindrical shock wave will grow with time t according to the equation

$$R = (\gamma + 1)^{1/2} \left(\frac{Q}{\pi\rho_0}\right)^{1/4} t^{1/2}$$

which is in full agreement with the law of propagation obtained by Lin (1954).

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