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GENERALIZED CONVEXITY IN MULTI-OBJECTIVE PROGRAMMING

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In this paper the relationships between solutions of (1) the weighting problem, (2) the kth objective Lagrangian problem, (3) the kth objective ϵ-constraint problem associated with a multi-objective nonlinear programming problem (MONLP) and noninferior solutions of the MONLP problem itself are studied under weaker types of convexities such as quasiconvexity, pseudoconvexity and η-convexity. The same ideas are applied to study refinement of duality theorems for MONLP. In particular the role played by linear approximation problem associated with MONLP has been discussed in detail. The concept of selective duality has also been investigated under weaker conditions of convexity.

1. INTRODUCTION

We consider the following multi-objective programming problem, also known as vector optimization problem (VOP) :

Given $f: R^N \to R^n$, $g: R^N \to R^m$, $S \subsetneq R^N$, find an $x^* \in X$ (if such an x^* exists) such that

$$f(x^*) = \min_{x \in X} f(x) \qquad \ldots(1.1.1)$$

where

$$X = \{x \in R^N : g(x) \leqq 0, x \in S\}. \qquad \ldots(1.1.2)$$

The following conventions for equalities and inequalities for vectors $x, y \in R^N$ are used (Mangasarian[12], p. 14):

$$x = y \Leftrightarrow x_i = y_i \text{ for all } i = 1, 2, \ldots, N;$$

$$x \geqq y \Leftrightarrow x_i \geqq y_i \text{ for all } i = 1, 2, \ldots, N;$$

$$x \geqslant y \Leftrightarrow x \geqq y \text{ but } x \neq y;$$

$$x > y \Leftrightarrow x_i > y_i \text{ for all } i = 1, 2, \ldots, N.$$

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A vector $x^* \in X$ is said to be a noninferior (or an efficient or a nondominated or a Pareto-Optimal) solution of the VOP if there exists no other feasible solution $x \in X$ such that $f(x) \leqslant f(x^*)$. We denote the set of all noninferior solutions of the VOP by X^*.

The most common strategy of getting the set X^* is to obtain nondominated solutions of the VOP in terms of optimal solutions of some appropriately associated scalar optimization problems. The following are the three common approaches of obtaining a scalar problem to be associated with VOP for this purpose.

(i) *The Weighting Problem*

Let $W = \{w \in R^n; w \geqq 0, \sum_{j=1}^{n} w_j = 1\}$ be the set of nonnegative weights in R^n. For a given $w \in W$ the weighting problem $P(w)$ is defined as

$$P(w): \min_{x \in X} \sum_{j=1}^{n} w_j f_j(x).$$

(ii) *The* k*th-objective Lagrangian Problem*

For a given k such that $1 \leqq k \leqq n$, the kth-objective Lagrangian problem is defined as

$$P_k(u): \min_{x \in X} \{f_k(x) + \sum_{\substack{j=1 \\ j \neq k}}^{n} u_j f_j(x)\}$$

where

$$u \in U_{(k)} = \{(u_1, u_2, \ldots, u_{k-1}, u_{k+1} \ldots, u_n)^T : u_j \geqq 0, \quad j = 1, 2, \ldots, n; j \neq k\} \subseteq R^{n-1}.$$

(iii) *The* k*th-objective* ϵ*-constraint Problem*

For a given k such that $1 \leqq k \leqq n$ and a given $\epsilon_{(k)} = (\epsilon_1, \ldots, \epsilon_{k-1}, \epsilon_{k+1}, \ldots, \epsilon_n)^T \in R^{n-1}$, the kth-objective ϵ constraint problem is defined as

$$P_k(\epsilon_{(k)}): \min_{x \in X} f_k(x)$$

subject to $f_j(x) \leqq \epsilon_j$, $j = 1, 2, \ldots, n$; $j \neq k$. For a given point $x^0 \in R^N$, the problem $P_k(\epsilon^0_{(k)})$ is defined as

$$P_k(\epsilon^0_{(k)}): \min_{x \in X} f_k(x)$$

subject to $f_j(x) \leqq \epsilon^0_j = f_j(x^0)$, $j = 1, 2, \ldots, n$; $j \neq k$.

That is $P_k\,(\epsilon^0_{(k)})$ is exactly $P_k\,(f_{(k)}\,(x^0))$.

Chankong and Haimes[8] (p. 119) have given some fundamental results concerning characterization of a noninferior solution of the VOP in terms of optimal solutions of these three scalarized optimization problems with the help of an implication diagram which involves the concept of convex functions. Hanson[5] and Kaul and Kaur[7] have defined new concepts of η-convex, η-strictly convex, η-pseudoconvex and η-quasiconvex functions which are weaker generalizations of the corresponding concepts of convex, strictly convex, pseudoconvex and quasiconvex functions.

In the second section of this paper we derive noninferior solutions of the VOP in terms of the three scalar optimization problems (SOP's) under weaker convexity assumptions. In the third section we study duality theory for the VOP under some weaker convexity assumptions, the pair of primal-dual multi-objective programs being that mentioned by Bitran[2].

2. Vector Optimization Theory

2.1. *Relationships Among the Three Forms of Scalarization*

First we consider the scalar nonlinear problem P : Given $f : R^N \to R$, $g : R^N \to R^m$, $S \subseteq R^N$, find an $x^* \in X$ (if such an x^* exists) such that

$$f\,(x^*) = \min_{x \in X} f(x) \qquad \ldots(2.1.1)$$

where

$$X = \{x \in R^N : x \in S,\, g\,(x) \leqq 0\}. \qquad \ldots(2.1.2)$$

The standard Lagrangian function $L\,(x, v)$ of the problem P is defined by

$$L\,(x, v) = f\,(x) + v^T\,g\,(x)$$

where $v \in R^m$.

Let S be an open set in R^N. Let $\bar{x}$ be a local minimum of P (cf. Mangasarian[12], p. 93). If g satisfies any of the constraint qualifications (for example, the Kunh-Tucker constraint qualification or the Weak Arrow-Hurwicz-Uzawa constraint qualification (see Mangasarian[12])), then there exists a $\bar{v} \in R^m$ such that $(\bar{x}, \bar{v})$ satisfies the Kuhn-Tucker conditions :

(a) $\nabla_{\bar{x}}\, L\,(\bar{x}, \bar{v}) = 0$

(b) $g\,(\bar{x}) \leqq 0$

(c) $\bar{v}^T\,g\,(\bar{x}) = 0$

(d) $\bar{v} \geqq 0$. $\qquad \ldots(2.1.3)$

When $x \in R^N$ and $u(x) \in R^n$, we shall use the symbol $\nabla_x u(x)$ to denote the $N \times n$ matrix, the jth column of which is the gradient of the component $u_j(x)$ of $u(x)$ w.r.t. x. Similar meaning is given to $\nabla_{\bar{x}} L(\bar{x}, \bar{v})$. For the sake of brevity, we shall also write $\nabla u(x)$ for $\nabla_x u(x)$.

Let I be the set of indices of active constraint functions at $\bar{x}$, i. e. $I = \{i : g_i(\bar{x}) = 0\}$. Let $I' = \{i : g_i(\bar{x}) < 0\}$. Let g_J. and g_I, be the subvectors of g corresponding to the index sets I and I', respectively. Let $v \in R^m$ be such that $v \geqq 0$ and $v^T g(\bar{x}) = 0$. Clearly $v_I \geqq 0$ and $V_I' = 0$ where v_I and $v_{I'}$ denote the subvectors of v corresponding to the index sets I and I', respectively. Let $K = \{i \in I : v_i = 0\}$ and $J = \{i \in I : v_i > 0\}$, and let g_J and g_K be the subvectors of g_I corresponding to the index sets J and K, respectively. The subvectors v_J and v_K of v_I are similarly defined. Now let $J1$, $J2$, $J3$ be any subsets of J such that $J1 \cup J2 \cup J3 = J$, $J1 \cap J2 = J2 \cap J3 = J3 \cap J1 = \phi$. Finally let v_{J_1}, v_{J_2}, v_{J3} and g_{J_1}, g_{J_2}, v_{J3} be the subvectors of v_J and g_J, respectively, corresponding to the sets $J1$, $J2$, $J3$.

The $(N + m)$ vector $(\bar{x}, \bar{v})$ which satisfies the Kuhn-Tucker condition (2.1.3), is said to satisfy the McCormick[13] second order sufficient optimality condition if f and g are twice differentiable and

$$\left.\begin{array}{l} d \in R^N \\ \nabla g_J(\bar{x})^T d = 0 \\ \nabla g_K(\bar{x})^T d \leqq 0. \end{array}\right\} \Rightarrow d^T \nabla_{\bar{x}\bar{x}} L(\bar{x}, \bar{v})\, d > 0. \qquad \text{...(2.1.4)}$$

The following theorem has been proved by Mahajan[11] in his unpublished Ph.D. thesis.

Theorem 2.1.1 [Mahajan[11], Theorem 1.4.2 p. 37]—Consider the scalar problem P defined by (2.1.1), (2.1.2). Let f and g be differentiable at $\bar{x} \in X$. If there exists a $v \in R^m$ such that $(\bar{x}, v)$ satisfies

(i) $(\nabla f(\bar{x}) + \nabla g(\bar{x})\, v)^T (x - \bar{x}) \geqq 0$ for all $x \in X$

(ii) $v \geqq 0$

(iii) $v^T g(\bar{x}) = 0$

and if, for any arbitrary disjoint subsets $J1$, $J2$, $J3$ of J whose union is J,

(iv) $f + v_{J1}^T g_{J_1}$ is pseudoconvex at $\bar{x}$ w.r.t. X,

(v) $v_{J_2}^T g_{J_2}$ is differentiable and quasiconvex at $\bar{x}$ w. r. t. X, and

(vi) g_{J3} is differentiable and quasiconvex at $\bar{x}$ w. r. t. X, then $\bar{x}$ is a global optimal solution of P.

Mahajan[11] has also pointed out three particular cases of the above result corresponding to

$$J1 = J,\ J2 = J3 = \phi \qquad \text{...(2.1.5a)}$$

$$J2 = J,\ J1 = J3, = \phi \qquad \text{...(2.1.5b)}$$

$$J3 = J,\ J1 = J2 = \phi. \qquad \text{...(2.1.5c)}$$

(See Mahajan[11], Remark 1 4.2, p. 39).

Define

$$f^T_{(k)} = (f_1, f_2, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n)^T.$$

Consider the problem $P_k\,(f_{(k)}\,(x^0))$ for a given point $x^0 \in X$. Under a suitable constraint qualification (for example the Kuhn-Tucker or the weak Arrow-Hurwicz-Uzawa constraint qualification given in Mangasarian[12]), the following Kuhn-Tucker conditions of optimality hold at a minimum x^* of $P_k\,(f_{(k)}\,(x^0))$.

$$\nabla f_k\,(x^*) + \nabla f_{(k)}\,(x^*)\,u + \nabla g\,(x^*)\,v = 0,\ u \in U_{(k)} \qquad \text{...(2.1.6a)}$$

$$v^T g\,(x^*) = 0 \qquad \text{...(2.1.6b)}$$

$$v_i \geqq 0,\ i = 1, 2, \ldots, m,\ u_j \geqq 0,\ j = 1, 2, \ldots, k-1, k, k+1, \ldots, n, \qquad \text{...(2.1.6c)}$$

$$g\,(x^*) \leqq 0.$$

Now we give the relations among the solutions of scalar optimization problems.

The following Lemma generalizes Lemma 4.6 of Chankong and Haimes[3] (p. 127) in the sense that convexity assumption is weakened.

Lemma 2.1.1—Let S be an open set in R^N. For a given $x^0 \in X$ let $x^* \in X$ solve $P_k\ (\epsilon^0_{(k)})$. Let f and g be differentiable at x^*. Assume that there exist $u \in U_{(k)}$ and a $v \in R^m$ such that (2.1.6) holds at (x^*, u, v).

Further assume that

(i) $f_k + u^T f_{(k)}$ is pseudoconvex at x^* w.r.t. X and g_J is quasiconvex at x^* w.r.t. X,

or

(ii) $f_k + u^T f_{(k)}$ is pseudoconvex at x^* w.r.t. X and $v_J^T\, g_J$ is quasiconvex at x^* w.r.t. X,

or

(iii) the numerical function $f_k + u^T f_{(k)} + v_J^T\, g_J$ is pseudoconvex at x^* w.r.t. X.

Then x^* solves $P_k(u)$.

PROOF : The lemma is an immediate consequence of the three cases of (2.1.5).

The proof of the following Lemma is trivial and is therefore omitted.

Lemma 2.1.2—Let S be an open set in R^N. For a given $x^0 \in X$ let x^* solve $P_k(\epsilon^0_{(k)})$. Let f and g be twice continuously differentiable at x^*. Let the Kuhn-Tucker constraint qualification (KTCQ) or the Weak-Arrow-Hurwicz-Uzawa constraint Qualification (WAHUCQ) hold at x^*. Let $U^0_{(k)} \subseteq U_{(k)}$ be such that for each $u \in U^0_{(k)}$ there exists a $v \in R^m$ such that (u, v) is a Kuhn-Tucker multiplier under the above C. Q. for $P_k(\epsilon^0_{(k)})$ and such that the following second order sufficient optimality condition holds for $P_k(u)$ at (u, v):

$$\left.\begin{array}{l} d \in R^N \\ \nabla g_J(x^*)^T d = 0 \\ \nabla g_K(x^*)^T d \leqq 0 \end{array}\right\} \Rightarrow d^T \nabla_{x^*x^*} L(x^*, u, v)\, d > 0 \qquad \ldots(2.1.7)$$

where $L(x^*, u, v) = f(x^*) + u^T f_{(k)}(x^*) + v^T g(x^*)$. Then x^* solves $P_k(u)$ for all $u \in U^0_{(k)}$.

Remark 2.1.1: Here we have imposed the second order sufficient optimality conditions in absence of convexity condition. The second order sufficient optimality condition automatically hold when $\nabla_{x^*x^*} L(x^*, u, v)$ is positive definite. Following is an example where convexity assumption does not hold for f but $\nabla_{x^*x^*} L(x^*, u, v)$ is positive definite.

minimize $(x^3, -x^3 + 2x, -4x)$

subject to $x^6 - 1 \leqq 0$.

One of the associated constraint problems for $x^0 = 0.8$ is

$P_1(\epsilon^0_{(1)})$: minimize x^3

$$\text{subject to } -x^3 + 2x \leqq 1.08$$
$$-4x \leqq 3.2$$
$$x^6 - 1 \leqq 1.$$

We find that $0.8 \leqq x \leqq 1$ are the feasible solutions and $x = 0.8$ solves $P_k(\epsilon^0_{(1)})$. $(0.8, u, v)$ where $u^T = (u_1, u_2)$, $u_1 \geqq 0$, $u_2 = \dfrac{1.92 + 0.08\, u_1}{4}$, $v = 0$, satisfies the Kuhn-Tucker conditions for $P_k(\epsilon^0_{(1)})$.

$$U^0_{(k)} = ((u_1, u_2) : 0 \leqq u_1 \leqq 1, u = \frac{1.92 + 0.08\, u_1}{4}).$$

$u \in U^0(k)$ and $v = 0$ are such that the second order sufficient optimality condition hold for $P_k(u)$, through f_1 is not convex. Hence 0.8 solves $P_k(u)$ for all $u \in U^0\ (k)$.

We now define η-convex, η-strictly convex, η-pseudoconvex and η-quasiconvex functions[5,7].

Definition 2.1.1—Let f be a numerical function defined on an open set $S \subseteq R^N$ and let f be differentiable at $x^0 \in S$. Then f is said to be

(i) η-convex at $x^0 \in S$ is $\exists$ a function $\eta : S \times S \to R^N$ such that, for all $x \in S$,

$$f(x) - f(x^0) \geqq \eta^T(x, x^0)\, \nabla f(x^0) \qquad \text{...(2.1.8)}$$

(ii) η-strictly convex at $x^0 \in S$ if $\exists$ a function $\eta : S \times S \to R^N$ such that strict inequality holds in (2.1.8) for all $x \in S$, $x \neq x^0$,

(iii) η-pseudoconvex at $x^0 \in S$ if $\exists$ a function $\eta : S \times S \to R^N$ such that

$$\eta^T(x, x^0)\, \nabla f(x^0) \geqq 0 \Rightarrow f(x) \geqq f(x^0) \text{ for all } x \in S \qquad \text{...(2.1.9)}$$

(iv) η-quasiconvex at $x^0 \in S$ if $\exists$ a function $\eta : S \times S \to R^N$ such that

$$f(x) \leqq f(x^0) \Rightarrow \eta^T(x, x^0)\, \nabla f(x^0) \leqq 0 \text{ for all } x \in S. \qquad \text{...(2.1.10)}$$

In addition we introduce the definition of η-strictly pseudoconvex function.

Definition 2.1.2—Let f be a numerical function on an open set $S \subseteq R^N$ and let f be differentiable at $x^0 \in S$. Then f is said to be η-strictly pseudoconvex at $x^0 \in S$ if $\exists$ a function $\eta : S \times S \to R^N$ such that for all $x \in S$, $x \neq x^0$

$$\eta^T(x, x^0)\, \nabla f(x^0) \geqq 0 \Rightarrow f(x) > f(x^0). \qquad \text{...(2.1.11)}$$

A function is said to have any of the above properties on S if it has that property at all points of S. By taking $\eta(x_1, x_2) = (x_1 - x_2)$ and by using the definition of Ponstein[14] it can be seen that every strictly pseudoconvex function is η-strictly pseudoconvex. The converse, however, is not true as can be seen from the following example (cf. Kaul and Kaur[7]).

Example —Let the function $f : S \to R$ be defined by $f(x) = x_1 + \sin x_2$. Let $S \subseteq R^2$ be given by

$$S = \{x \in R^2 : 4x_1^2 + 4x_2^2 - 9 \leqq 0,\ x_1, x_2 \geqq 0\}$$

and let the function $\eta : S \times S \to R^2$ be given by

$$\eta(x, u) = \begin{pmatrix} \dfrac{\sin x_1 - \sin u_1}{\cos u_1} \\ \dfrac{\sin x_2 - \sin u_2}{\cos u_2} \end{pmatrix}.$$

Then the function $f: S \to R$ defined by $f(x) = x_1 + \sin x_2$ is η-strictly pseudoconvex on S, but it is not strictly pseudoconvex on S since at $x = (\frac{\pi}{3}, 0)$, $u = (\frac{\pi}{6}, \frac{\pi}{3})$ we have $(x - u)^T \nabla f(u) = 0$ whereas $f(x) < f(u)$. Kaul and Kaur[7] have used the function in the above example to show that exery η.pseudoconvex function need not be pseudoconvex.

We define a function to be η-concave, η-strictly concave, η-pseudoconcave, η-strictly pseudoconcave, η-quasiconcave, at $x^0 \in S$ (on S) if and only if $-f$ is η-convex, η-strictly convex, η-pseudoconvex, η-strictly pseudoconvex, η-quasiconvex at $x^0 \in S$ (on S) respectively.

The following Lemma is similar to Theorem 2.1.1.

Lemma 2.1.3—Let $\bar{x} \in S$. Assume that there evists a $\bar{v} \in R^m$ such that $(\bar{x}, \bar{v})$ safisfies the Kuhn-Tucker conditions (2.1.3). Further assume that

(i) f is η-pseudoconvex at $\bar{x}$ and g_J is η-quasiconvex at $\bar{x}$ for the same function η.

or

(ii) f is η-pseudoconvex at $\bar{x}$ and $\bar{v}_J^T g_J$ is η-quasiconvex at $\bar{x}$ for the same function η

or

(iii) $f + \bar{v}_J^T g_J$ is η-pseudoconvex at $\bar{x}$.

Then $\bar{x}$ is an optimai solution of problem P defined by (2.1.1) and (2.1.2).

The Lemma stated below is similar to Lemma 2.1.1 above.

Lemma 2.1.4—Let x^* solve P_k $(\epsilon^0_{(k)})$ and let (x^*, u, v), $u \in U_{(k)}$ $v \in R^m$ satisfy the Kuhn-Tucker conditions (2.1.6) of optimality. Let

(i) $f_k + u^T f_{(k)}$ be η-pseudoconvex at x^* and g_J be η-quasiconvex at x^* for the same function η,

or

(ii) $f_k + u^T f_{(k)}$ be η-pseudoconvex at x^* and $v_J^T g_J$ be η-quasiconvex at x^* for the same function,

or

(iii) $f_k + u^T f_{(k)} + v_J^T g_J$ be η-pseudoconvex at x^*. Then x^* solves $P_k(u)$.

Let (x^*, u, v), $u \in U_k$, $v \in R^m$ satisfy the Kuhn-Tucker conditions (2.1.6) of optimality for $P_k(\epsilon^0_{(k)})$. Then (2.1.6a) can be written in the form

$$\sum_{j=1}^{n} u'_j \nabla f_j (x^*) + \sum_{i=1}^{m} v_i \nabla g_i (x^*) = 0$$

where

$$u'_j = u_j \text{ for } j \neq k, u'_k = 1.$$

Dividing throughout by $\sum_{j=1}^{n} u'_j$ and writing

$$w_j = u'_j \left(\sum_{j=1}^{n} u'_j \right)^{-1} \text{ and } \bar{v}_i = \bar{v}_i \left(\sum_{j=1}^{n} u'_j \right)^{-1}$$

(2.1.6a) can now be written as

$$\sum_{j=1}^{n} w_j \nabla f_j (x^*) + \sum_{i=1}^{m} \bar{v}_i \nabla g_i (x^*) = 0. \quad \text{...(2.1.12a)}$$

Moreover, we note that $w_j \geqq 0$ and $\sum_{j=1}^{n} w_j = 1$.

Similarly (2.1.6b) through (2.1.6d) can be written as

$$\bar{v}_i^T g_i (x^*) = 0 \quad i = 1, 2, .. , m, \quad \text{...(2.1.12b)}$$

$$\bar{v}_i \geqq 0 \quad i = 1, 2 ..., m, \quad \text{...(2.1.12c)}$$

$$g_i (x^*) \leqq 0 \quad i = 1, 2, ..., m. \quad \text{.. (2.1.12d)}$$

This leads to the following lemma which is a generalization of Lemma 4.1 of Chankong and Haimes[3].

Lemma 2.1.5—Let x^* solve $P_k(\epsilon^0_{(k)})$ and let $(x^*, \bar{v}, w)$, $\bar{v} \in R^m$, $w \in W$ satisfy (2.1.12). Assume that

(i) $w^T f$ is pseudoconvex (η-pseudoconvex) at $x^* \in S$ and g_J is quasiconvex (η-quasiconvex for the same function η) at x^*.

or

(ii) $w^T f$ is pseudoconvex (η-pseudoconvex) at $x^* \in S$ and $\bar{v}_J^T g_J$ is quasiconvex (η-quasiconvex for the same function η) at x^*

or

(iii) $w^T f + \bar{v}_J^T g_J$ is pseudoconvex or η-pseudoconvex at x^*. Then x^* solves $P(w)$.

2.2. *Characterizing Noninferior Solutions in Terms of the Solutions of the kth-Objective ε-Constraint Problem.*

We now establish the relationship between the noninferior solutions of vector optimization problem and the kth-objective ϵ-constraint problem.

Theorem 2.2.1 Let S be an open set in R^N. Let x^* solve $P_k\ (\epsilon^*_{(k)})$. Let $f_{(k)}$ be η-quasiconvex at x^* and f_k be η-strictly pseudoconvex and η-quasiconcave at x^* for the same function η. Then $x^* \in X^*$.

PROOF : Since x^* solves $P_k\ (\epsilon^*_{(k)})$

$$f_k(x^*) = \min_{x \in X} f_k(x) \qquad \ldots(2.2.1)$$

$$\text{subject to } f_{(k)}(x) \leqq f_{(k)}(x^*). \qquad \ldots(2.2.2)$$

Since f_k is η-quasiconcave at x^* from (2.2.1)

$$\eta^T(x, x^*)\, \nabla f_k\ (x^*) \geqq 0, \text{ for all } x \in X \qquad \ldots(2.2.3)$$

and $f_{(k)}$ is η-quasiconvex at x^* for the same function η,

$$\eta^T(x, x^*)\, \nabla f_{(k)}(x^*) \leqq 0.$$

Hence,

$$\left.\begin{array}{l} \eta^T(x, x^*)\, \nabla f_k(x^*) < 0 \\ \eta^T(x, x^*)\, \nabla f_{(k)}(x^*) \leqq 0 \end{array}\right] \text{has no solution } x \in X. \qquad \ldots(2.2.4)$$

Suppose $x^* \notin X^*$. Then there exists an $\hat{x} \in X$, $\hat{x} \neq x^*$ such that $f(\hat{x}) \leqslant f(x^*)$ and since f_k is η-strictly pseudoconvex at x^* and $f_{(k)}$ is η-quasiconvex at x^*

$$\eta^T(\hat{x}, x^*)\, \nabla f_k(x^*) < 0$$

$$\eta^T(\hat{x}, x^*)\, \nabla f_{(k)}(x^*) \leqq 0$$

contradicting (2.2.4). Hence $x^* \in X^*$.

The following corollary is an immediate consequence.

Corollary 2.2.2—Let S be an open set in R^N. Let x^* solve $P_k\ (\epsilon^*_{(k)})$. Let $f_{(k)}$ be quasiconvex at x^* and f_k be strictly pseudoconvex and quasiconcave at x^*. Then $x^* \in X^*$.

The well known results regarding the noninferiority for vector optimization problem of solutions of the kth-objective constraint problem are the following (cf. Lin[9,10], Chankong and Haimes[3]) :

(i) If x^* solves $P_k(\epsilon^*_{(k)})$ for some k and if the solution is unique, then x^* is a noninferior solution of VOP.

(ii) If x^* solves $P_k(\epsilon^*_{(k)})$ for every $k = 1, 2, \ldots, n$ then $x^* \in X^*$.

(iii) Let x^* solve $P_k(\epsilon^*_{(k)})$ for some k. For this k, let Y_k be the set of all $\epsilon_{(k)} \in R^{n-1}$ such that $P_k(\epsilon_{(k)})$ is feasible. Let $\phi_k(\epsilon_{(k)}) = \inf\{f_k(x) : x \in X, f_j(x) \leqq \epsilon_j$ for each $j \neq k\}$ and let $\hat{Y}_k = \{\epsilon_{(k)} : \epsilon_{(k)} \in Y_k, \phi_k(\epsilon_{(k)}) > -\infty$ and there exists an $x^0 \in X$ such that $f_k(x^0) = \phi_k(\epsilon_{(k)})\}$. Then $x^* \in X^*$ iff $\phi_k(\epsilon_{(k)}) > \phi_k(\epsilon^*_{(k)})$ for all $\epsilon_{(k)} \in \hat{Y}_k$ such that $\epsilon_{(k)} \leqq \epsilon^*_{(k)}$.

To generate noninferior solutions using the results (i) to (iii) is not very practical. For instance, in (i), (ii) and (ii) the vector x^* which appears in the constrainrs of $P_k(\epsilon^*_{(k)}$ is assumed to be its solution which is rather difficult to comprehend. For a convex problem $P_k(.)$, strict convexity of the (primary) objective function $f_k(.)$ guarantees uniqueness without further checking. The above theorem shows that if the (primary) objective function $f_k(.)$ is strictly pseudoconvex as well as quasiconcave at the point x^* and the other objective functions are quasiconvex at x^* then it guarantees noninferiority.

As an example we see that the function $f(x) = -\cos^2(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is both strictly pseudoconvex and quasiconcave at the point $x^* = 0$.

2.3. *Characterizing Noninferior Solutions in Terms of the Solutions of the Weighting Problem*

The following theorem is a generalization of Theorem 4.5 of Chankong and Haimes.[3].

Theorem 2.3.1—Let S be an open set in R^N. Let $x^* \in X^*$ and let $x^*, \bar{v}, w)$ satisfy (2.1.12). Assume that

(i) $w^T f$ is pseudoconvex (η-pseudoconvex) at x^* and g_J is quasiconvex (η-quasiconvex for the same function η) at x^*,

or

(ii) $w^T f$ is pseudoconvex (η-pseudoconvex) at x^* and $\bar{v}_J^T g_J$ is quasiconvex (η-quasiconvex for the same function η) at x^*,

or

(iii) $w^T f + \bar{v}_J^T g_J$ is pseudoconvex (η-pseudoconvex at x^*). Then x solves $P(w)$.*

PROOF : Since $x^* \in X^*$ it solves $P_k (\epsilon^*_{(k)})$ for all $k = 1, 2, ..., n$ (Chankong and Haimes[3], Theorem 4.1). Now the result follows from Lemma 2.1.5.

The following theorem gives the conditions under which a solution of the weighting problem is a noninferior solution of the vector optimization problem.

Theorem 2.3.2—Let S be an open set in R^N. Let x^* solve $P(w)$. Assume that g satisfies the KTCQ or WAHUCQ at x^*. Let $w_k \neq 0$ and the corresponding f_k be strongly quasiconvex (cf. Bazaraa and Shetty[1]) and pseudoconvex at x^*, $f_{(k)}$ and g be quasiconvex at x^*. Then $x^* \in X^*$.

PROOF : The vector g satisfies the KTCQ or WAHUCQ at $x^* \in X$; so there exists a $v \in R^m$ such that

$$\sum_{i=1}^{n} w_j \nabla f_j (x^*) + \sum_{i=1}^{m} v_i \nabla g_i (x^*) = 0;$$

$$\sum_{i=1}^{m} v_i g_i (x^*) = 0,$$

$$v_i \gneqq 0$$

$$g_i (x^*) \leqq 0, \; i = 1, 2, ..., m,$$

or, since $w_k \neq 0$, for appropriate w_j' a dn $_i'$,

$$\nabla f_k (x^*) + \sum_{\substack{j=1 \\ j \neq 1}}^{n} w_j' \nabla f_j (x) + \sum_{i=1}^{m} v_i' \nabla g_i (x^*) = 0.$$

These are the Kuhn-Tucker conditions of optimality for $P_k (\epsilon^*_{(k)})$. We have f_k pseudoconvex at x^* and g and $f_{(k)}$ quasiconvex at x^*. Then by sufficient optimality theorem (Mangasarian[12], Theorem 10.1.1) x^* is a solution of $P_k (\epsilon^*_{(k)})$.

Also, the set $x^2 = \{x \in R^N : g(x) \lneqq 0, f_{(k)}(x) - f_{(k)}(x^*) \leqq 0\}$, of feasible solution of $P_k (\epsilon^*_{(k)})$ is convex set in R^N and f_k is strongly quasiconvex at x^*. Hence x^* is the unique solution of $P_k (\epsilon^*_{(k)})$ (Bazarra and Shetty[1], Theorem 3.5.9). We now apply the well the known result (Theorem 4.2, p. 129, Chankong and Haimes[3]) that if x^* solves $P_k (\epsilon^*_{(k)})$ for some k and if the solution is unique, then x^* is noninferior solution of the VOP.

The well known conditions under which x^* is a non-inferior solution of the VOP when it solves $P(w)$ are (Chankong and Haimes[3], Theorem 4.6):

(i) $w_j > 0$ for all $j = 1, 2, \quad , n$

or

(ii) x^* is the unique solution of $P(w)$.

In the above theorem we have shown the noninferiority with some weaker convexity assumption without assuming uniqueness condition or the condition that all the weights are positive.

It has been shown that a function is strictly pseudoconvex if it is strongly quasiconvex and Pseudoconvex (Ponstein[14]). There are, however, functions which are strongly quasiconvex but not pseudoconvex and vice versa. For example, the function $f(x) = x + x^3, x \in R$ is strictly pseudoconvex every-where. The function $f(x) = x^3$ is strongly quasiconvex at $x = 0$ but is not pseudoconvex at $x = 0$. The function $f(x) = \sin x + \cos x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is not strongly quasiconvex at $x = 0$, but is pseudoconvex at $x = 0$.

Theorem 2.3.3—Let S be an open set in R^N. Let x^* solve $P(w)$ for a given $w \in W$. Assume that $W^T f$ is η-quasiconcave at x^* and the function f is η-strictly pseudoconvex at x^* for the same function η. Then x^* is a noninferior solution of the VOP.

PROOF : Since x^* solves $P(w)$, $w^T f(x) \geqq w^T f(x^*)$ for all $x \in X$ and the function $w^T f$ is η-quasiconcave at x^* which implies

$$\eta^T (x, x^*) \nabla f(x^*) w \geqq 0 \text{ for all } x \in X. \qquad \ldots(2.3.!)$$

Suppose $x^* \notin X^*$. Then there exists an $\hat{x} \in X$, $\hat{x} \neq x^*$ such that $f(\hat{x}) \leqslant f(x^*)$, and since f is η-strictly pseudoconvex at x^*, this implies that

$$\eta^T (x, x^*) \nabla f(x^*) < 0$$

and since $w \in W$

$$\eta^T (x, x^*) \nabla f(x^*) w < 0.$$

This contradicte (2.3.1). Hence $x^* \in X^*$.

As an immediate consequence, we get :

Corollary 2.3.4—Let S be an open set in R^N. Let x^* solve $P(w)$ for a given $w \in W$. Assume that $w^T f$ is differentiable and quasiconcave at x^* and the function f is strictly pseudoconvex at x^*. Then $x^* \in X^*$.

Remark 2.3.1: Theorem 2.3.3 also holds, if instead of the assumption that the function f is η-strictly pseudoconvex at x^*, either of the following assumptions is made:

(i) The components of f corresponding to the non-zero components of w are η-strictly pseudoconvex at x^*.

(ii) $w^T f$ is η-strictly pseudoconvex at x^*.

Remark 2.3.2: For example consider the VOP

$$\text{minimize } f^T(x) = (f_1(x), f_2(x), f_3(x))$$
$$= (-\cos x, \sin^2 x, x^3 + \sin x)$$

where the domain of definition of all the function is

$$X = \{x \in R : -\frac{\pi}{4} \leqq x \leqq \frac{\pi}{4}\}.$$

Take $w^T = (\frac{1}{2}, \frac{1}{2}, 0)$. Then it is easy to check that the objective function of the associated weighting problem $w^T f(x) = \frac{1}{2}(\sin^2 x - \cos x)$ is quasiconcave at $x^* = 0$ and the objective function $f(x)$ of the VOP is strictly pseudoconvex at $x^* = 0$. Moreover x^* solves the weighting problem $P(w)$. Hence $0 \in X^*$.

The apparently anomalous situation arising due to the assumption that x^* is involved in the definition of $P_k(\epsilon^*_{(k)})$ and at the same time solves $P_k(\epsilon^*_{(k)})$ made in Theorem 2.2.1 and the results stated thereafter, can be overcome to some extent through the following result.

Result 1—Let x^* solve $P_k(\epsilon^0_{(k)})$ for some given $x^0 \in X$ and let $(x^*, \bar{v}, w)$, $\bar{v} \in R^m$, $w \in W$ satisfy the Khun-Tucker conditions (2.1.12). Let

(i) $w^T f$ be η-pseudoconvex at x^* and g_J be η-quasiconvex at x^* for the same function η,

or

(ii) $w^T f$ be η-pseudoconvex at x^* and $\bar{v}_J^T g_J$ be η-quasiconvex at x^* for the same function η,

or

(iii) $w^T f + \bar{v}_J^T g_J$ be η-pseudoconvex at x^*.

Further assume that

(1) f_k is η-strictly pseudoconvex at x^*, $f_{(k)}$ and g are η-quasiconvex at x^*, for the same function η,

or

(2) $w^T f$ is η-quasiconcave at x^* and f is η-strictly pseudoconvex at x^* for the same function η.

Then $x^* \in X^*$.

PROOF : From the first set of assumptions we get x^* solves $P(w)$ and from the second set of assumptions we get $x^* \in x^*$.

2.4. *Characterizing Noninferior Solutions in Terms of the Solutions of the Lagranian Problem*

The following theorem is generalization of Theorem 4.7 in Chankong and Haimes[3] in the sense that convexity assumptions are weakened.

Theorem 2.4.1—Let S be an open set in R^N. Let $x^* \in X^*$ and assume that there exist a $u \in U_{(k)}$ and a $v \in R^m$ such that (2.1.6) holds at (x^*, u, v). Further assume that

(i) $f_k + u^T f_{(k)}$ is pseudoconvex (η-pseudoconvex) at x^* and g_J is quasiconvex (η-quasiconvex for the same function η) at x^*,

or

(ii) $f_k + u^T f_{(k)}$ is pseudoconvex (η-pseudoconvex) at x^* and $v_J^T g_J$ is quasiconvex (η-quasiconvex for the same function η) at x^*,

or

(iii) $f_k + u^T f_{(k)} + v_J^T g_J$ *is* pseudoconvex or η-pseudoconvex at x^*,

or

(iv) the second order sufficient optimality conditions (2.1.7) hold good.

Then x^* solves $P_k(u)$.

PROOF : Since (x^*, u, v) satisfies (2.1.6), by applying the three cases of (2.1.5), and Lemma 2.1.2 and Lemma 2.1.4 we get this theorem.

We prove the following theorems in the reverse direction which are generalizations of Theorem 4.8 given in Chankong and Haimes[3].

Theorem 2.4.2—Let S be an open set in R^N. Let x^* solve $P_k(u)$ for some $u \in U_{(k)}$. Assume $f_{(k)}$ is η-quasiconvex at x^* and f_k is η-strictly pseudoconvex and η-quasiconcave at x^* for the same function η. Then $x^* \in X^*$.

PROOF : Since x^* solves $P_k(u)$ for some $u \in U_{(k)}$, x^* also solves P_k $(\epsilon^*_{(k)})$ (Chankong and Haimes[3], Lemma 4.7), and by Corollary 2.2.2, x^* is a noninferior solution of the VOP.

Theorem 2.4.3—Let S be an open set in R^N. Let $x^* \in X$ solves $P_k(u)$ for some $u \in U_{(k)}$. Let g satisfy the KTCQ or WAHUCQ at x^*. Let f_k be strictly pseudoconvex at x^*, $f_{(k)}$ and g be quasiconvex at x^*. Then $x^* \in X^*$.

PROOF : Since x^* solves $P_k(u)$, it solves $P(w)$ with

$$w_j = u_j \left(1 + \sum_{j=1}^{n} u_j\right)^{-1}, \; j = 1, 2, \ldots, n; j \neq k,$$

$$w_k = \left(1 + \sum_{j=1}^{n} u_j\right)^{-1}$$

Then from Theorem 2.3.2, we get the result.

Theorem 2.4.4—Let S be an open set in R^N. Let x^* solve $P_k(u)$ for some $u \in U_{(k)}$. Let

$$w_j = u_j \left(1 + \sum_{j=1}^{n} u_j\right)^{-1}, \; j = 1, 1, \ldots, n, \; j \neq k,$$

$$w_k = \left(1 + \sum_{j=1}^{n} u_j\right)^{-1}$$

Let $w^T f$ be quasiconcave (η-quasiconcave) at x^* and f be strictly pseudoconvex (η-strictly pseudoconvex for the same function η) at x^*. Then $x^* \in X^*$.

PROOF : Since x^* solves $P_k(u)$ it solves $P(w)$ with the given values of w_j, $j = 1, 2, \ldots, n$ and then the theorem follows from Theorem follows from Theorem 2.3.3.

Remark similar to Remark 2.3.1 for Theorem 2.3.3 also applies to Theorem 2.4. 4 above.

We now give a result similar to Result 1, Section 2.3.

Result 2—For a given $x^0 \in X$ let x^* solve $P_k(\epsilon^0_{(k)})$ and let (x^*, u, v) satisfy (2.1.6). Assume that

(i) $f_k + u^T f_{(k)}$ is η-pseudoconvex at x^* and g_J is η-quasiconvex at x^* for the same function η,

or

(ii) $f_k + u^T f_{(k)}$ is η-pseudoconvex at x^* and v_J^T is η-quasiconvex at x^* for the same function η,

or

(iii) $f_k + u^T f_{(k)} + v_J^T$ is η pseudoconvex at x^*.

Further assume that $f_{(k)}$ is η-quasiconvex at x^* and f_k is η-strictly pseudoconvex at x^* and η-quasiconcave at x^* for the same function η. Then $x^* \in X^*$.

PROOF : From the first set of assumptions we get x^* solves $P_k(u)$ and from the next assumption we get x^* is a noninferior solution of the VOP.

2.5. *Kuhn-Tucker Necessary and Sufficient Conditions for Noninferiority*

Denfiition—A feasible point x^* for VOP ((1.1.1), (1.1.2)) is said to satisfy the Kuhn-Tucker conditions for noninferiority (KTCN) for the VOP if

(i) all f and g are differentiable and $S \neq \phi$

and

(ii) there exist $\lambda \in R^n$ and $\mu \in R^m$ such that $\lambda \geqslant 0$

$$\mu \geqq 0, \sum_{j=1}^{n} \lambda_j \nabla F_j(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0,$$

and

$$\mu_i g_i(x^*) = 0; \ i = 1, 2, \ldots, m.$$

The following theorem generalizes Theorem 4.10 given in Chankong and Haimes[3] in the sense that the convexity conditions are weakened.

Theorem 2.5.1—Let x^* satisfy the KTCN for VOP ((1.1.1), (1.1.2)). Assume g and f are quasiconvex and for at least one k for which $\lambda_k > 0$ in the KTCN the corresponding f_k is strictly pseudoconvex at x^*. Then $x^* \in x^*$,

PROOF : Since x^* satisfies the KTCN for the VOP

$$\sum_{j=1}^{n} \lambda_j \nabla f_j(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0.$$

Let $\lambda_k > 0$ and let the corresponding f_k be strictly pseudoconvex at x^*. Then the above equation can be rewritten as

$$\nabla f_k(x^*) + \sum_{\substack{j=1 \\ j \neq 1}}^{n} \lambda_j^0 \nabla f_j(x^*) + \sum_{i=1}^{m} \mu_i^0 \nabla g_i(x^*) = 0$$

where $\lambda_j^0 \geqq 0$ and $\mu_i^0 \geqq 0$ for $j = 1, 2, \ldots, n$; $j \neq k$ and $i = 1, 2, \ldots, m$. This together with other conditions above are equivalent to the Kuhn-Tucker conditions for optimality of P_k ($\epsilon^*_{(k)}$). The rest of the argument is similar to that in the proof of Theorem 2.3.2 (See Remarks in the paragraph immediately preceding theorem 2.3.3).

2.6. *Sufficient Conditions for Proper Noninferiority*

Geoffrion[4] reformulated the definition of proper noninferiority given by Kuhn-Tucker[8]. The following is the definition given by Geoffrion[4].

Definition —A noninferior solution of the VOP, x^*, is said to be a proper noninferior solution if there exists a scalar $M > 0$ such that for each i, $i = 1, 2, \ldots, n$ and each $x \in X$ satisfying $f_i(x) < f_i(x^*)$ there exists at least one $j \neq i$ with $f_j(x) > f_j(x^*)$ and

$$\frac{f_i(x) - f_i(x^*)}{f_j(x^*) - f_j(x)} \leqq M.$$

We have the following theorem on proper noninferiority which is a slight generalization of Theorem 4.13b of Chankong and Haimes[3].

Theorem 2.6.1—Let S be an open set in R^N. Let $x^* \in S$ solve $P_k(\epsilon^0_{(k)})$ and let $(x^*, \bar{v}, w)$ satisfy the Kuhn-Tucker conditions (2.1.12) and let the Kuhn-Tucker multipliers w_j associated with the constraints $f_j(x) \leqq f_j(x^0), j \neq k$, be strictly positive. Let

(i) $w^T f$ be pseudoconvex (η-pseudoconvex) at x^* and g_J be quasiconvex (η-quasiconvex for the same function η) at x^*,

or

(ii) $w^T f$ be pseudoconvex (η-pseudoconvex) at x^* and $\bar{v}_J^T g_J$ be quasiconvex (η-quasiconvex for the same, function η) at x^*,

or

(iii) $w^T f + \bar{v}_J^T g_J$ be pseudoconvex or η-pseudoconvex at x^*. Then x^* is a proper noninferior solution.

PROOF : From Lemma 2.1.5 it follows that x^* solves $P(w)$. We have the Kuhn-Tucker multipliers w_j to be strictly positive. The theorem now follows from well known result of Geoffrion[4] that if x^* solves $P(w)$ for some $w > 0$, then x^* is a proper noninferior solution of the VOP.

3. Duality Theory in Nonlinear Multiobjective Programming

Let Γ be the set of $m \times n$ real matrices. Consider the set of ordered pairs of vectors (η, ξ), $\eta \in R^n$, $\xi \in R^m$, defined by

$F = \{(\eta, \xi) \in R^{n+m} : \eta \geqq f(x), \xi \geqq g(x)$, for some $x \in R^N\}$, and the multiple criteria optimization problem (MCOP)

$$D(\pi) : EF(D(\pi)) = \min_{(\eta,\xi)\in F} \{\eta + \pi^T \xi\}$$

where $\pi \in \Gamma$, and $EF(D(\pi))$ denote the set of noninferior or efficient values of the objective function of Problem $D(\pi)$ Let $S = R^N$. Let

$$Y = \bigcup_{\pi \in \Gamma} EF(D(\pi)).$$

The dual problem to the VOP ((1 1.1), (1.1.2)) is defined to be MCOP

$$(D) : EF(D) = \underset{\rho \in Y}{\text{Max}} \; \rho$$

where $EF(D)$ denotes the set of noninferior or efficient values of the objective function of Problem (D).

We adopt the following conventions.

(a) For a minimization [maximization] MCOP we define EF (MCOP) $= (-\infty, \ldots, -\infty)$ $[(+\infty, \ldots; +\infty)]$ whenever for every $x^* \in X$ there exists, a sequence $\{x^j\}_{j=1}^{\infty}$, with $x^j \in X$ such that

$$f(x^{j+1}) \leqslant f(x^j) \leqq f(x^*) \quad [f(x^{j+1}) \geqslant f(x^j) \geqq f(x^*)]$$

and for some $k \in \{1, 2, \ldots, n\}$

$$\lim_{j \to \infty} f_k(x^j) = -\infty \; [+\infty].$$

(b) Whenever the feasible set of minimization [maximization] MCOP is empty, we define

$$EF(\text{MCOP}) = (+\infty, \ldots, +\infty) \; [(-\infty, \ldots, -\infty)].$$

A matrix π is said to be feasible in Problem (D) if $EF(D(\pi)) \neq (-\infty, \ldots, -\infty)$.

The following results are given by Bitran[2].

(3A) For any feasible matrix π in Problem (D) there is an $w \in R^n$ such that $w > 0$, $\pi w \geqq 0$.

(3B) Let x^* be feasible in the VOP ((1.1.1), (1.1.2)) $f(x^*) \in EF(D(\pi))$ for some $\pi \in \Gamma$. Then x^* is a noninferior solution of the VOP.

(3C) Assume that $x^* \in X = \{x \in R^N : g(x) \leqq 0\}$ is a noninferior solution of the following linear approximation multiple criteria optimization problem (LAP) to the VOP at x^*.

$$(\text{LAP}) : \text{Min} \{x^T \nabla f(x^*) : -(x - x^*)^T \nabla g_I(x^*) \geqq 0\}$$

where $I = \{i : g_i(x^*) = 0\} = \{1, 2, \ldots, q\}$ say; $I \subseteq \{1, 2, \ldots, m\}$.

The index set I corresponds to the constraints active at x^*. Then there exist $q \times n$ matrices U^1, U^2, U^*, with

$$U^* = U^1 + U^2$$

such that

$$\lambda^T [\nabla f(x^*) + \nabla g_I(x^*) U^*] - t'^T u^* \leqslant 0$$

$$\lambda \in R^N, \; t' \in R^q, \; t' \geqq 0 \qquad \ldots(3.1.1)$$

has no solution and

$$-x^{*T} \nabla f(x^*) = x^{*T} \nabla g_I(x^*) U^1 + e_1^{0^T} U^2 \quad ...(3.1.2)$$

where $e_1^0 \in R^q$ and $e_1^{0^T} = (1, 0, ..., 0)$.

The above result 3.C has been given by Bitran[2] as a consequence of a similar result obtained by Isermann[6] for MOLPP.

It may be noted that construction of (LAP) at x^* such that x^* itself is its noninferior solution may pose a separate problem.

Bitran[2] has also extended the result so as to include, the constraints nonactive at x^*,

Let $I' = \{i : g_i(x^*) < 0\}$ be the index set corresponding to the constraints nonactive at x^*.

Define

$$\pi^i = \binom{U^i}{0};\ i = 1, 2, * \quad ...(3.1.3)$$

where the zero matrix in π^i is of order $(m-q) \times n$ and corresponds to the constraints nonactive at x^*. Clearly π^1, π^2, π^* are $m \times n$ matrices. From the relation between U^*, U^1 and U^2, it follows that $\pi^* = \pi^1 + \pi^2$.

We can now rewrite (3.1.1) and (3.1.2), respectively, as

$$\lambda^T [\nabla f(x^*) + \nabla g(x^*) \pi^*] + t^T \pi^* \leqslant 0$$

$$\lambda \in R^N, t \in R^m, t \geqq 0 \quad ...(3.1.4)$$

has no solution and

$$-x^{*^T} \nabla f(x^*) = x^{*^T} \nabla g(x^*) \pi^1 + e_1^I \pi^2 \quad ...(3.1.5)$$

$$e_1 \in R^m, e_1^T = 1, 0, ..., 0.$$

The above result can be stated in a more general setting as follows :

Let $x^* \in X = \{x \in R^N : g(x) \leqq 0\}$ be a noninferior solution to the following LAP associated with the VOP.

$$(\text{LAP})^* : \text{Min}\ \{x^T \nabla f(x) : -(x - x^*)^T \nabla g(x^*) \geqq 0\}.$$

Then there exist $m \times n$ matrices π^1, π^2 and π^* such that $\pi^* = \pi^1 + \pi^2$ and such that

$$\lambda^T [\nabla f(x^*) + \nabla g(x^*) \pi^*] + t^T \pi^* \leqslant 0,$$

$$\lambda \in R^N, t \in R^m, t \geqq 0 \quad ...(3.1.6)$$

has no solution and

$$-x^{*^T} \nabla f(x^*) = x^{*^T} \nabla g(x^*) \pi^1 + \alpha^T \pi^2 \qquad ...(3.1.7)$$

for some $\alpha \in R^m$.

We now state the following :

Weak Duality Theorem[2]—For any matrix π feasible in Problem (D) and any x feasible in the VOP, we have

$$f(x) \nleq z \text{ for all } z \in EF(D(\pi)).$$

Combining the result (3A), (3B) and the Weak Duality Theorem above, Bitran[2] has given the following important result.

(3D) Let x be feasible in the VOP, $f(x) \in EF(D(\pi))$ for some $\pi \in \Gamma$. Then $f(x) \in EF(D)$, x is a noninferier solution of the VOP and π is a noninferior solution of Problem (D).

3.1. *Relationships Among the Solution of the* (LAP), *Noninferior Solution of the* VOP, *and Noninferior Solution of Problem* (D)

We now prove the following theorem with the notation and results developed above.

Theorem 3.1.1—If a noninferior solution x^* of the linear approximation (LAP) MCOP to the VOP at x^* is feasible for the VOP, f is pseudoconvex and g_I is differentiable and quasiconvex at x^*, then $x^* \in X^*$.

PROOF : Assume $x^* \notin X^*$. Then there exists an $\hat{x} \in X$, $\hat{x} \neq x^*$ such that $f(\hat{x}) \leqslant f(x^*)$. Since f is pseudoconvex at x^*, this implies

$$(\hat{x} - x^*)^T \nabla f(x^*) \leqslant 0.$$

Since $x^* \in X$, $g_I(\hat{x}) \leqq g_I(x^*) = 0$, and further since g_I is quasiconvex at x^*,

$$(\hat{x} - x^*)^T \nabla g_I(x^*) \leqq 0.$$

So

$$\hat{x} \in \{x \in R^N : -(x - x^*)^T \nabla g_I(x^*) \geqq 0 \text{ and}$$
$$(x - x^*)^T \nabla f(x^*) \leqslant 0\}.$$

This contradicts the statement that x^* is a noninferior solution of (LAP). Consequently every point $\xi \in X$, which is noninferior solution of (LAP) at ξ itself, is a noninferior solution of the VOP.

Remark : The above theorem is an improvisation of earlier known result [Bitran[2], Proposition 2.6 (i)], where the VOP is a maximization problem and f and g

are assumed to be concave and differentiable on R^N. Here we have weakened the convexity/concavity assumptions.

We now give the converse of the above theorem.

Theorem 3.1.2—Let x^* be a noninferior solution of the VOP. Let f be pseudoconcave and g be pseudoconcave at x^*. Then x^* is a noninferior solution of the (LAP)*:

$$(\text{LAP})^* : \text{Min } \{x^T \nabla f(x^*) : -(x - x^*)^T \nabla g(x^*) \geqq 0\}.$$

PROOF : Assume that x^* is not a noninferior solution of (LAP)*; then there exists an $\hat{x} \neq x^*$ such that

$$\hat{x}^T \nabla f(x^*) \leqslant x^{*^T} \nabla f(x^*)$$

i. e. $$(\hat{x} - x^*)^T \nabla f_i(x^*) \leqslant 0 \qquad \text{...(3.1.8)}$$

$$\text{subject to } (\hat{x} - x^*)^T \nabla g(x^*) \leqq 0.$$

Since g is pseudoconcave at x^*, this implies $g(\hat{x}) \leqq g(x^*) \leqq 0$, i. e. $\hat{x} \in X$.

Since f is pseudoconcave and hence quasiconcave, for all components i of the L.H.S. of the vector inequality (3.1.8) for which $(\hat{x} - x^*)^T \nabla f_i(x^*) < 0$ holds, we shall have $f_i(\hat{x}) < f_i(x^*)$. On the other hand for all other components j we shall have from the same vector inequality $(\hat{x} - x^*)^T \nabla f_j(x^*) = 0$. Since f is pseudoconcave, $f_j(\hat{x}) \leqq f_j(x^*)$.

Combining these we shall have

$$f(\hat{x}) \leqslant f(x^*). \qquad \text{...(3.1. 9)}$$

But (3.1.9) contradicts that x^* is a noninferior solution of the VOP. Hence x^* is a noninferior solution of the (LAP)*.

Let x^* be a noninferior solution of (LAP)*. Then from result (3C) it follows

$$\lambda^T [\nabla f(x^*) + \nabla g(x^*) \pi^*] + t^T \pi^* \leqslant 0$$
$$\lambda \in R^N, \ t \in R^m, t \geqq 0 \qquad \text{...(3.1.6)}$$

has no solution and

$$-x^{*^T} \nabla f(x^*) = x^{*^T} \nabla g(x^*) \pi^1 + \alpha^T \pi^2, \text{ for some } \alpha \in R^m. \qquad \text{...(3.1.7)}$$

By applying Motzkin's theorem of the alternative[12] to (3.1.6) it follows that there exists a $w_0 \in R^n$, $w_0 > 0$ such that

$$[\nabla f(x^*) + \nabla g(x^*) \pi^*] w_0 = 0 \qquad ...(3.1.10)$$

$$\pi^* w_0 \geqq 0. \qquad ...(3.1.11)$$

First, we note from the definition of π^* in (3.1.3), that

$$\pi^*_{ji} g_i(x^*) = 0, \; j = 1, 2, ..., m;$$

$$i = 1, 2, ..., n. \qquad ...(3.1.11a)$$

We shall now make use of w_0 determined in (3.1.10) in the following theorem.

Theorem 3.1.3—Assume that x^* is a noninferior solution of (LAP)* such that $x^* \in X$. Assume that $w_0^T [f + \pi^{*T} g]$ is η-pseudoconvex at x^* where π^* is defined in (3.1.3). Then (i) $f(x^*) \in EF(D(\pi^*))$ (ii) $f(x^*) \in EF(D)$ (iii) π^* is a noninferior solution of Problem (D).

PROOF : Note that, by definition,

$$D(\pi^*) = \min_{(\eta, \xi) \in F} \{\eta + \pi^{*T} \xi\}.$$

Assume that $f(x^*) \notin EF(D(\pi^*))$. Then there exists an $\bar{x} \in R^N$, $(\bar{\eta}, \bar{\xi}) \in R^{n+m}$ such that

$$\bar{\eta} \geqq f(\bar{x}), \; \bar{\xi} \geqq g(\bar{x}) \text{ and}$$

$$\bar{\eta} + \pi^{*T} \bar{\xi} \leqslant f(x^*) = f(x^*) + \pi^{*T} g(x^*) \qquad ...(3.1.12)$$

since $\pi^{*T} g(x^*) = 0$.

Multiplying (3.1.12) on the left by w_0^T, and noting $w_0 > 0$, we get

$$w_0^T \bar{\eta} + w_0^T \pi^{*T} \bar{\xi} < w_0^T f(x^*) + w_0^T \pi^{*T} g(x^*).$$

Also, from $\bar{\eta} \geqq f(\bar{x})$ and $\bar{\xi} \geqq g(\bar{x})$, we get

$$w_0^T f(\bar{x}) + w_0^T \pi^{*T} g(\bar{x}) \leqq w_0^T \bar{\eta} + w_0^T \pi^{*T} \bar{\xi}.$$

Hence

$$w_0^T f(\bar{x}) + w_0^T \pi^{*T} g(\bar{x}) < w_0^T f(x^*) + w_0^T \pi^{*T} g(x^*). \qquad ...(3.1.13)$$

Since $w_0^T\ [f + \pi^{*T} g]$ is η-pseudoconvex at x^* from (3.1.10) we get

$$w_0^T f(\bar{x}) + w_0^T \pi^{*T} g(\bar{x}) \geqq w_0^T f(x^*) + w_0^T \pi^{*T} g(x^*).$$

This contradicts (3.1.13).

Therefore $f(x^*) = f(x^*) + \pi^{*T} g(x^*) \in EF(D(\pi^*))$. This proves (i), (ii), (iii) follows from result 3. D.

Remark 3.1.1 : Bitran[2] (Proposition 2.6 (ii)) considered a similar problem of maximization where f and g were assumed to be concave and differentiable on R^N. In the Theorem 3.1.3 above it has been found enough to assume $w_0^T\ [f + \pi^{*T} g]$ to be η-pseudoconvex at x^*. In this sense Proposition 2.6 (ii) of Bitran[2] is a particular case of Theorem 3.1.3 above.

Remark 3.1.2 : Theorems similar to Theorems 3.1.1—3.1.3 can be formulated in a straight forward manner so as to apply to situations where not all components of the constraint function g are differentiable.

3.2. *Direct Duality*

We begin the discussion of direct duality theory with the following theorem.

Theorem 3.2.1 Let x^* be a noninferior solution of the VOP. Assume that (x^*, v, w), $v \in R^m$, $w \in R^n$ satisfies

$$\nabla f(x^*)\ w + \nabla g(x^*)\ v = 0 \quad \text{...(3.2.1a)}$$

$$v^T g(x^*) = 0 \quad \text{...(3.2.1b)}$$

$$v \geqq 0,\ w \geqslant 0,\ \sum_{i=1}^{n} w_i = 1. \quad \text{...(3.2.1c)}$$

Assume that $w^T f + v^T g$ is η-strictly yseudoconvex at x^*. Then there exists a $\pi^0 \in \Gamma$ such that $f(x^*) \in EF(D(\pi^0))$ and $\pi^0 \in EF(D))$.

PROOF : In the vector w some of the component may be zero. Define π^0 as the $m \times n$ matrix whose ith column is equal to zero or v according as $w_i = 0$ or $w_i \neq 0$. Then $\pi^0 w = v$. Assume that $f(x^*) \notin EF(D(\pi^0))$. Then there exists $(\eta, \xi) \in F$ such that

$$\eta + \pi^{0T} \xi \leqslant f(x^*).$$

Hence

$$w^T\eta + \pi^T \pi^{0T} \xi = w^T \eta + v^T \xi \leqslant w^T f(x^*)$$

The rest of the proof of this theorem is similar to that of Theorem 3.1.3.

We now use the concept of selective duality and obtain the following direct duality theorem. Assume that a nonempty subset Δ of $\{1, 2, ..., n\}$ is such that for each $j \in \Delta$, f_j is differentiable at x^*. Similarly, suppose a nonempty subset Δ' of $\{1, 2, ..., m\}$ is such that for each $i \in \Delta'$, g_i is differentiable at x^*. Let f_Δ, $g_{\Delta'}$ be the subvectors of f and g corresponding to the index sets Δ and Δ' respectively.

Theorem 3.2.2—Let x^* be a noninferior solution of the VOP. Assume that $(x^*, v_\Delta, w_\Delta)$ satisfies

$$\nabla f_\Delta (x^*) w_\Delta + \nabla g_{\Delta'} (x^*) v_{\Delta'} = 0 \qquad ...(3.2.2a)$$

$$v^T_{\Delta'} g_{\Delta'} (x^*) = 0 \qquad ...(3.2.2b)$$

$$v_{\Delta'} \geqq 0, w_\Delta \geqslant 0, \sum_{i \in \Delta} (w_\Delta)_i = 1. \qquad ...(3.2.2c)$$

Assume that $w^T_\Delta f_\Delta + v^T_{\Delta'} g_{\Delta'}$ is η-strictly pseudoconvex at x^*. Then there exists a $\pi^0 \in \Gamma$ such that $f(x^*) \in EF(D(\pi^0))$ and $\pi^0 \in EF(D)$.

PROOF : Define $v \in R^m$, $v = \begin{bmatrix} v_\Delta \\ 0 \end{bmatrix}$ where the $(m - |\Delta'|)$- zero vector corresponds to the constraint functions nondifferentiable at x^*. Also define

$w = \begin{bmatrix} w_\Delta \\ 0 \end{bmatrix} \in R^n$ where the $(n - |\Delta|)$-zero vector corresponds to the objective functions nondifferentiable at x^*.

Define π^0 as the $m \times n$ matrix with the ith column equal to zero or v according as $w_i = 0$ or $w_i \neq 0$.

Assume $f(x^*) \notin EF(D(\pi^0))$. Then there exists $(\eta, \xi) \in F$ such that

$$\eta + \pi^{0T} \xi \leqslant f(x^*)$$

and $\eta \geqq f(\bar{x}), \xi \geqq g(\bar{x})$ for some $\bar{x} \in R^N$.

Hence

$$w^T \eta + w^T \pi^{0T} \xi \leqq w^T f(x^*),$$

which implies

$$w^T \eta + v^T \xi \leqq w^T f(x^*).$$

This further gives

$$w^T f(\bar{x}) + v^T g(\bar{x}) \leqq w^T \eta + v^T \xi \leqq w^T f(x^*)$$

from which we get

$$w_{\Delta}^{T} f_{\Delta}(\bar{x}) + v_{\Delta'}^{T} g_{\Delta'}(\bar{x}) \leqq w_{\Delta}^{T} f_{\Delta}(x^*) + v_{\Delta'}^{T} g_{\Delta'}(x^*)$$

and as $w_{\Delta}^{T} f_{\Delta} + v_{\Delta'}^{T} g_{\Delta'}$ is assumed to be η-strictly pseoudoconvex at x^* this implies

$$\eta^T(\bar{x}, x^*)[\nabla f_{\Delta}(x^*) w_{\Delta} + \nabla g_{\Delta'}(x^*) v_{\Delta'}] < 0.$$

But (3.2.2a) implies

$$\eta^T(\bar{x}, x^*)[\nabla f_{\Delta}(x^*) w_{\Delta} + \nabla g_{\Delta'}(x^*) v_{\Delta'}] = 0$$

which is a contradiction. Hence $f(x^*) \in EF(D(\pi^0))$ and also $f(x^*) \in EF(D)$ follows from result 3D.

Remark 3.2.2 : As a special case of the above theorem we obtian the usual direct duality theorem.

3.3. *Saddle Point Duality, Stability and Kuhn-Tucker Conditions.*

Definition 3.3.1—A pair (x^*, π^*) is said to be a saddle point of the vector valued Lagrangian

$$L(\eta, \xi, \pi) = \eta + \pi^T \xi$$

if

$$f(x^*) + \pi^T g(x^*) \not\geq f(x^*) + \pi^{*T} g(x^*) \not\geq \eta + \pi^{*T} \xi \qquad \ldots(3.3.1)$$

for all $(\eta, \xi) \in F$ and all π such that

$$\pi^T \xi \leqslant 0, \ \xi \geqq 0,$$

has no solution.

Definition 3.3.2—The perturbation point-to-set map $v : R^m \to \mathcal{P}(R^n)$ is defined as

$$v(y) = EF(P_y)$$

where $EF(P_y)$ is defined through the problem (P_y) given by

$$(P_y): EF(P_y) = \min\{f(x) : x \in X_y\}$$

$$X_y = \{x \in R^N : g(x) \leqq y\}.$$

That is, $v(y)$ is the image of $EF(X_y)$ under $f(\,.\,)$. Here the constraints $g(x) \leqq 0$ are perturbed to $g(x) \leqq y$, the vector y being the perturbation. Clearly, therefore, $X = X_0$.

Definition 3.3.3—The VOP is said to be M-stable at $x^* \in X$ if there is a $n \times m$ matrix M such that

$$v(y) \cap \{(f(x^*) - My) + R_{\geqq}^{n}\} = \phi \text{ for all } y \in R^m \qquad \ldots(3.3.2)$$

where

$$R^n_- = \{x \in R^n : x \leqslant 0\}.$$

Definition 3.3.4—Kuhn-Tucker conditions :

The Kuhn-Tucker conditions for a pair (x^*, π^*) with $x^* \in R^N$ and $\pi^* \in \Gamma$ can be stated as follows :

$$\left.\begin{array}{l} \lambda^T [\nabla f(x^*) + \nabla g(x^*) \pi^*] + t^T \pi^* \leqslant 0 \\ \lambda \in R^N, t \in R^m, t \geqq 0 \end{array}\right\} \text{has no solution} \quad \text{.. (3.3.3a)}$$

$$\pi^{*T} g(x^*) = 0; \quad \text{...(3.3.3b)}$$

$$g(x^*) \leqq 0. \quad \text{...(3.3.3c)}$$

The Kuhn-Tucker conditions (3.3.3) can also be stated as follows :

$$[\nabla f(x^*) + \nabla g(x^*) \pi^*] w_0 = 0, \; w_0 \geqslant 0,$$

$$\pi^* w_0 \geqq 0 \quad \text{.. (3.3.4)}$$

$$\pi^{*T} g(x^*) = 0$$

$$g(x^*) \leqq 0.$$

The next theorem gives the relation among the Kuhn-Tucker conditions, a noninferior solution of the VOP and a noninferior solution of Problem (D). We state it without proof since it is a paraphrase of Theorem 3.1.3 and result (3D). This theorem is an improvisation of Proposition 3.6 of Bitran[2] in the sense that convexity assumptions are weakened.

Theorem 3.3.1—Assume that the Kuhn-Tucker conditions (3.3.4) hold at (x^*, π^*, w_0) and $w_0^T [f + \pi^{*T} g]$ is η-pseudoconvex at x^*. Then $f(x^*) \in EF(D(\pi^*))$, x^* is a noninferior solution of the VOP, $f(x^*) \in EF(D)$ and x^* is a noninferior solution of Problem (D).

The Kuhn-Tucker conditions for the case when objective functions and constraint functions are not all differentiable can be formulated as follows. Let $\Delta, \Delta', f_\Delta, g_\Delta$, have the same meaning as defined earlier.

Write

$$\pi^* = \begin{array}{c} \\ |\Delta'| \\ m - |\Delta'| \end{array} \begin{array}{c} \begin{array}{cc} |\Delta| & n - |\Delta| \end{array} \\ \begin{bmatrix} \pi^*_{\Delta'\Delta} & 0 \\ 0 & 0 \end{bmatrix} \end{array}. \quad \text{...(3.3.5)}$$

The Kuhn-Tucker conditions corresponding to (3.3.4) can now be stated as follows.

$$[\nabla f_{\Delta}(x^*) + \nabla g_{\Delta'}(x^*)\, \pi^*_{\Delta'\Delta}]\, w_{\Delta} = 0,\ w_{\Delta} > 0,$$

$$\pi^*_{\Delta'\Delta}\quad w_{\Delta} \geqq 0,$$

$$\pi^*_{\Delta\Delta}\ g_{\Delta'}(x^*) = 0 \qquad \ldots(3.3.6)$$

$$g_{\Delta'}(x^*) \leqq 0.$$

We now state a theorem involving LAP's associated with the VOP where not all components of f and g are differentiable.

Theorem 3.3.2—If $x^* \in X$ solves the following linear approximation problem (LAP_{Δ}) to the VOP at x^* :

$$(LAP_{\Delta}) : \text{Min}\ \{x^T \nabla f_{\Delta}(x^*) : -(x - x^*)^T \nabla g_{I_{(\Delta')}}(x^*) \geqq 0$$

then there is a $|\Delta'| \times |\Delta|$ matrix $\pi^*_{\Delta'\Delta}$ and w_{Δ} such that $(x^*, \pi^*_{\Delta'\Delta}, w_{\Delta})$ satisfies the Kuhn-Tucker conditions (3.3.6).

The following theorem is straight forward extension of Proposition 3.5 of Bitran[2].

Theorem 3.3.4—Assume that the Kuhn-Tucker conditions (3.3.4) hold at a pair (x^*, π^*) and $w_0^T\ [f + \pi^{*T} g]$ is η-pseudoconvex at x^*, then (i) the VOP is M-stable at x^* with $M(x^*) = \pi^*$ and (ii) (x^*, π^*) is a saddle point of $L(\eta, \xi, \pi)$

Theorem 3.3.5—Let (i) (x^*, π^*) be a saddle point of $L(\eta, \xi, \pi)$ or (ii) the VOP be M-stable at x^* with $M(x^*) = \pi^*$ and $\pi^{*T} g(x^*) = 0$. Let f be pseudoconcave and let the constraint g bo pseudoconcave at x^*. Then the Kuhn-Tucker condition (3.3.5) hold at the pair (x^*, π^*).

Proof : From Proposition 3.5 of Bitran[2] we have x^* is a noninferior solution of the VOP. Then from Theorem 3.1.2 we have x^* is a noninferior solution of (LAP)* of Theorem 3.1.2, and consequently (x^*, π^*) satisfies the Kuhn-Tucker conditions (3.3.4).

The following theorem gives the conditions under which a solution of the weighting problem $P(w)$ satisfies the Kuhn-Tucker conditions (3.3.4).

Theorem 3.3.6—Let x^* solve $P(w_0)$ where $w_0 > 0$. Assume w_0^T is quasiconcave and differentiable at x^* and g_I is quasiconvex and differentiable at x^*. Then x^* satisfies the Kuhn-Tucker conditions (3.3.4).

PROOF : Since $w_0^T f$ is quasiconcave and differentiable at x^* and g_I is quasiconvex and differentiable at x^*, we get x^* solves the linear approximation to problem $P(w_0)$ at x^*, i. e.,

$$\text{Min } \{x^T \nabla f(x^*) w_0 : -(x - x^*)^T \nabla g_I(x^*) \geqq 0\}.$$

Suppose x^* does not solve the linear approximation to the VOP at x^* i. e. there exists a $\hat{x} \in X$, $\hat{x} \neq x^*$ such that

$$\hat{x}^T \nabla f(x^*) \leqslant x^{*T} \nabla f(x^*)$$

or

$$(\hat{x} - x^*)^T \nabla f(x^*) \leqslant 0.$$

As $w_0 > 0$ this implies $(\hat{x} - x^*)^T \nabla f(x^*) w_0 < 0$.

This contradicts that x^* solves the linear approximation to Problem $P(w_0)$ at x^*. Hence x^* solves the linear approximation to the VOP at x^*. This implies there is $\pi \in \Gamma$ such that (x^*, π^*) satisfies the Kuhn-Tucker conditions (3.3.4).

CONCLUDING REMARKS

In this paper an attempt has been made to investigate the extent to which convexity/concavity assumptions on the objective functions or constraint functions in a VOP can be relaxed to weaker assumptions of pseudoconvexity/pseudoconcavity, quasiconvexity/quasiconcavity, etc. The problems of necessary and sufficient optimality conditions, some aspects of duality theory such as direct duality and selective duality, have been analyzed with this idea in mind. It seems the converse duality theory is not amenable to such processes of weakening the underlined convexity/concavity assumption.

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