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AN OPTIMAL PROGRAMME FOR AUGMENTATION OF CAPACITIES OF DEPOTS AND SHIPMENT OF BUSES FROM DEPOTS TO STARTING POINTS OF ROUTES

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The problem of determination of an optimal programme for augmentation of capacities of depots and the number of buses to be parked overnight at respective depots and their shipment from depots to the starting points of routes is considered. The objective is to minimize the total capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost of travel performed by buses between the depots and the starting points of the routes over a planning horizon. A method is developed to obtain the solution of this problem. Further, a software package for the method is developed and tested on ICL 2900 Computer.

1. INTRODUCTION

Problems relating to urban bus transportation system have been of interest for quite some time.

In the past, these problems have been tackled by applying rudimentary techniques based on commonsense and experience. This has resulted at times into avoidable expenses. On account of this and growing complexity of problems relating to urban bus transportation system and increasingly larger sums of money involved, the need to employ analytical tools to deal with these problems has been lately felt. In recent times, analytical tools have been attempted to deal with them by several researchers[1-6].

Buses are parked overnight at depots. Starting points of routes are generally different from depots and a bus has to travel from its depot to the starting point of its route before it can be engaged on regular service. Similarly, at the end of the service, the bus travels back to its depot. Distance traversed by a bus in going from its depot to the starting point of its route and back pays no return and is known as dead mileage or dead-travelling. It is desirable to reduce dead-travelling. On the other hand, depots built in the past cannot provide parking facilities to the increased number of buses necessitated by an increase in the number of city-bus travellers. Capacities of the depots have to be augmented to satisfy the enhanced demand of parking facilities. Capital expenditure to be incurred in creating capacity for a bus varies from depot to depot. There is an upper bound on the number of buses by which capacity at each depot can be augmented. The problem is to determine the number of buses by which capacities of different depots should by augmented and the number of buses to be parked overnight at them and the number of buses to be sent from each depot to the starting point of each route with an objective to minimize the total capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost associated with the dead-travelling of buses over the planning horizon. This problem is formulated as a capacitated transportation-type problem and is solved by the procedure given by Dantzig[2] after making some alterations. The solution procedure is illustrated through a numerical example and tested on ICL 2900 Computer. Utility and versatility and future extension of the present work are also discussed.

2. Formulation of the Problem

Suppose that there are m existing depots, p sites for potential depots and n routes. Let the existing capacity measured in terms of buses at depot i ($i = 1, \ldots, m + p$) be a_i. For a potential depot i ($i = m + 1, \ldots, m + p$), the existing capacity a_i is equal to zero. Let b_j ($j = 1, \ldots, n$) be the current enhanced demand of bused on route j, c the cost in rupees of running a bus per kilometre, $c_{i\,(n+1)}$ ($i = 1, \ldots, m + p$) the expenditure in multiple of thousand rupees to be incurred in augmenting capacity for one bus at depot i, d_{ij} ($i = 1, \ldots, m + p; j = 1, \ldots, n$) the distance in kilometres between depot i and the starting point of route j, $f_{i(n+1)}$ ($i = 1, \ldots, m + p$) the upper bound on augmentation of capacity of depot i, N the planning horizon in years, r the annual rate of interest, x_i ($i = 1, \ldots, m + p$) the number of buses to be parked overnight at depot i, x_{ij} ($i = 1, \ldots, m + p; j = 1, \ldots, n$) the number of buses to be sent from depot i to the starting point of route j, $x_i\,(n + 1)$ ($i = 1, \ldots, m + p$) the augmented capacity at depot i. The objective is to minimize the total capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost associated with the dead-travelling of buses between the depots and the starting points of the routes over the planning horizon. The mathematical formulation of the problem is as follows. Find integer x_i and $x_{ij} \geqslant 0$ ($i = 1, \ldots, m + p, j = 1, \ldots, n + 1$) which minimize

$$Z = \sum_{i=1}^{m+p} \sum_{j=1}^{n+1} c_{ij}\, x_{ij} \qquad \ldots(1)$$

where

$$c_{ij} = \left[365\ (2)\ d_{ij} \sum_{t=1}^{N} (1 + r/100)^{-t} \right] 10^{-3}$$

$$(i = 1, \ldots, m + p; j = 1, \ldots, n) \qquad \ldots(2)$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} - x_{i\,(n+1)} = a_i \quad (i = 1, \ldots, m + p) \qquad \ldots(3)$$

$$\sum_{i=1}^{m+p} x_{ij} = b_j \quad (j = 1, \ldots, n) \qquad \ldots(4)$$

$$x_i = \sum_{j=1}^{N} x_{ij}\ (i = 1, \ldots, m + p) \qquad \ldots(5)$$

and the upper bound restrictions

$$x_{i\,(n+1)} \leqslant f_{i(n+1)}\ (i = 1, \ldots, m + p). \qquad \ldots(6)$$

It may be noted that c_{ij} $(i = 1, \ldots, m + p; J = 1, \ldots, n)$ in multiple of thousand rupees is the present value of the cost associated with the dead-travelling of a bus between depot i and the starting point of route j over the planning horizon. Equation (5) are the consequence of the fact that the number of buses to be parked at depot i should be equal to the number of buses being sent from depot i to the starting points of the various routes.

3. Solution Procedure

The problem formulated above can obviously be solved using the branch and bound method of integer programming. But as this method involves much computational work, an efficient method based on the procedure for solving the capacitated transportation problem is developed.

Note that eqns. (5) are not active constraints of the formulated problem. So an optimal solution of the formulated problem can be obtained ignoring eqns. (5). The optimal solution of the formulated problem ignoring eqns. (5) would determine required values of all x_{ij}'s. After the values of the x_{ij}'s are determined, eqns. (5) are used to compute required values of all x_i's. As the formulated problem ignoring eqns. (5) possesses total unimodularity for its coefficient matrix and a_i's and b_j's are integers, integrality of all x_{ij}'s of its basic feasible solutions is guaranteed. Unimodularity and its implications are discussed by Bazaraa and Jarvis[1]. With the above observation, the formulated problem ignoring eqns. (5) is a capacitated transportation-type problem which differs from the usual capacitated transportation problem considered by Dantzig[2] in that the coefficients of $x_i\,(n+1)$ $(i = 1, \ldots, m+p)$ in constraints (3) are -1 in it, whereas the coefficients of all the variables appearing in the constraints are 1 in the usual capacitated transportation problem. To obtain the optimal solution of the formulated problem ignoring eqns. (5), work proceeds as follows.

To overcome the difficulty in obtaining an initial basic feasible solution of the formulated problem ignoring eqns. (5), we introduce artificial integer variables $x_{(m+p+1)j} \geqslant 0$ $(J = 1, \ldots, n)$ into eqns. (4) which then assume the form

$$\sum_{i=1}^{m+p+1} x_{ij} = b_j \; (j = 1, \ldots, n). \qquad \ldots(7)$$

The upper bound restrictions on aritficial variables are

$$x_{(m+p+1)j} \leqslant \infty \; (J = 1, \ldots n). \qquad \ldots(8)$$

Entry of artificial variables into optimal solution is prevented by associating a cost M with each of them where M is an arbitrarily large positive number. After this, the formulated problem ignoring eqns. (5) reduces to the following augmented problem which requires determining integer $x_{ij} \geqslant$ $(i = 1, \ldots, m+p+1; j = 1, \ldots, n+1)$; but not $i = m+p+1$ and $J = n+1$ simultaneously) that minimize

$$Z' = \sum_{i=1}^{m+p} \sum_{j=1}^{n+1} c_{ij}\, x_{ij} + M \sum_{j=1}^{n} x_{(m+p+1)j} \qquad \ldots(9)$$

subject to the constraints (3) and (7) and upper bound restrictions (6) and (8). The augmented problem also possesses total unimodularity for its coefficient matrix and so integrality of all x_{ij} 's of its basic feasible solutions is guaranteed. Solution of the augmented problem provides the soultion of the formulated problem ignoring eqns. (5). The tableau representation of the augmented problem is shown in Table I. In this Table, the left top corner of each cell $(m+p+1, j)$ in the row with the heading "Artificial variable" depicts the cost associated with the artificial variable $x\,(m+p+1)\,j$, while the left top corner of each cell $(i, n+1)$ in the column with the heading "Augmented capacity" depicts the expenditure to be incurred in augmenting capacity for one bus at depot i. And the left top corner of each other cell (i, j) in the tableau depicts the present value of the cost associated with the dead-travelling of a bus between depot i and the starting point of route j over the planning horizon. The left middle space of each cell $(i, n+1)$ in the column with the heading "Augmented capacity" depicts the upper bound on augmentation of the capacity of depot i. And the left middle space of all other cells depicts an entry of ∞ indicating that there is

TABLE I

Augmented problem

	Starting points of route 1	route 2	…	route n	Augmented capacity $x_{i(n+1)}$	Existing capacity a_i
Depot 1	c_{11} ∞ 1	c_{12} ∞ 1	…	c_{1n} ∞	$c_{1(n+1)}$ f_1 -1	a_1
Depot 2	c_{21} ∞ 1	c_{22} ∞ 1	…	c_{2n} ∞ 1	$c_{2(n+1)}$ $f_{2(n+1)}$ -1	a_2
	⋮	⋮	…	⋮	⋮	⋮
Depot $(m+p)$	$c_{(m+p)}^{1}$ ∞ 1	$c_{(m+p)}^{2}$ ∞ 1	…	$c_{(m+p)n}$ ∞ 1	$c_{(m+p)(n+1)}$ $f_{(m+p)(n+1)}$ -1	a_{m+p}
Artificial variable $x_{(m+p+1)j}$	M ∞ 1	M ∞ 1	…	M ∞ 1		
Buses required b_j	b_1	b_n		b_n		

no upper bound restriction on values of the variables corresponding to them. To obtain a_i in the row with the heading 'Depot i', sum the products obtained by multiplying x_{ij} with the entry in the left bottom corner of the corresponding cell across the row and to obtain b_j in the column with the subheading "route j", sum the x_{ij}'s acoss the column. And Z' is obtained by summing the products obtained by multiplying x_{ij} with the entry in the left top corner of the corresponding cell all over the tablaeu.

An initial basic feasible solution for the augmented problem is found almost in the same way as for the standard cost minimizing transportation problem. Several methods for obtaining an initial basic feasible solution for the standard cost minimizing transportation problem are discussed by Hadley[3]. Any of these methods after some modification can be used to obtain an initial basic feasible solution for the augmented problem. In the case of the standard cost minimizing transportation problem, all the methods for determining an initial basic feasible solution assign a nonnegative value to a variable and at the same time satisfy either a row or a column constraint at each step. But in the case of the augmented problem, a nonnegative value is assigned to a variable without forcing it to exceed its upper bound and at the same time satisfying either a row or a column constraint. If the value of the variable is limited by a row or a column constraint, the variable is considered basic. If, on the other hand, the value of the variable is limited by its upper bound, it is considered nonbasic. However, if the value of the variable is limited by its upper bound and also by a row or a column constraint, it is considered basic. It is worth noting that x_{ij} multiplied by the entry in the left bottom corner of the corresponding cell (i, j) is the amount used of resource a_i and x_{ij} alone is the amount satisfied of requirement b_j in the case of the augmented problem, whereas x_{ij} is the amount used of resource a_j and satisfied of requirement b_j in the case of the standard cost minimizing transportation problem. The value of a basic variable is entered in the right bottom corner of the associated cell after enclosing it in a circle. Also the value of a nonbasic variable different from zero is entered in the right bottom corner of the associated cell but after making a bar over it.

After a basic feasible solution has been obtained, it is tested for optimality. To do this, the relative cost coefficients $\bar{c}_{ij}$'s corresponding to the nonbasic cells are computed and their values are entered in the right middle spaces of these cells. To compute the $\bar{c}_{ij}$'s we proceed as follows. The $(m + p)$ u_i 's and nj 's are computed from the following three different forms of equations

$$\left.\begin{array}{ll} u_i + v_j = c_{ij} & \text{corresponding to the basic cells in rows with the heading of depots and columns with the subheading of routes} \\ u_i = -c_i\,(n + 1) & \text{corresponding to the basic cells in the column with the heading of augmented capacity} \\ v_j = M & \text{corresponding to the basic cells in the row with the heading of artificial variable} \end{array}\right\} \quad \ldots(10)$$

Once the $(m + p)$ u_i's and n v_j's are known, $\bar{c}_{ij}$'s are computed from the following three different forms of equations

$$\left.\begin{array}{ll} \bar{c}_{ij} = c_{ij} - u_i - v_j & \text{corresponding to the nonbasic cells in rows with the heading of depots and columns with the subheading of routes} \\ \bar{c}_{i\,(n+1)} = c_{i(n+1)} + u_i & \text{corresponding to the nonbasic cells in the column with the heading of augmented capacity} \\ \bar{c}_{i\,(m+p+1)j} = M - v_j & \text{corresponing to the nonbasic cells in the row with the heading of artificial variable} \end{array}\right\} \quad \ldots(11)$$

If all the relative cost coefficients corresponding to the nonbasic variables at zero level are nonnegative and at their upper bound are nonpositive, the basic feasible solution is optimal. On the other hand, if at least one of the relative cost coefficients corresponding to the nonbasic variables at zero level is negative or at their upper bounds is positive, the basic feasible solution is not optimal. Then the nonbasic variable with which is associated the greatest numerical value of the relative cost coefficient among all the nonbasic variables at zero level with negative values or at their upper bounds with positive values of the relative cost coefficients is selected to enter the basis of the new feasible solution. Further, if the nonbasic variable selected to enter the basis is at zero level, $\theta \geqslant 0$ is added to its value and values of the basic variables are so adjusted that all the row and column constraints are satisfied. Then a value as large as possible is assigned to θ so that the value of no variable exceeds its upper bound. This yields a new basic feasible solution. And if the nonbasic variable selected to enter the basis is at its upper bound, $\theta \geqslant 0$ is subtracted from its value and then the same procedure is followed as in the case of the nonbasic variable at zero level selected to enter the basis. Irrespective of what type of nonbasic variable enters the basis, the leaving basic variable is chosen as the one which reaches either a zero value or a value equal to its upper bound. To ensure unique selection of leaving variable, that variable among the potential leaving variables is chosen as the leaving variable for which row index i is minimun. In case, there are two or more potential leaving variables with the smallest row index, then the one with the smallest column index j is chosen as the leaving variable. The new basic feasible solution thus obtained is again tested for optimality and the procedure is repeated till the optimal solution is obtained. Finally, it may be mentioned that if the optimal solution of the augmented problem has one or more artificial variables with positive values, then the formulated problem has no feasible solution.

4. A Numerical Example

Now we shall apply the above procedure to obtain the optimal solution of a numerical problem which is obtained by taking $m = 2, p = 1, n = 4, c = 4$, $d_{11} = 5, d_{12} = 7, d_{13} = 3, d_{14} = 4, d_{21} = 8, d_{22} = 9, d_{23} = 10, d_{24} = 4, d_{31} = 6$, $d_{32} = 2, d_{33} = 8, d_{34} = 6, N = 10, r = 10$, and assigning numerical values to all other quantities in the problem formulated above in section 2. The tableau representation of the augmented problem associated with the numerical problem is shown in Table II. For this augmented problem, the objective function which we seek to minimize is

$$Z' = \left\{ \begin{array}{l} 90x_{11} + 126\, x_{12} + 54\, x_{13} + 72\, x_{14} + 630\, x_{15} \\ + 144\, x_{21} + 161\, x_{22} + 179\, x_{23} + 72\, x_{24} + 550\, x_{25} \\ + 108\, x_{31} + 36\, x_{32} + 144\, x_{33} + 108\, x_{34} + 610\, x_{35} \\ + \quad M(x_{41} + \quad x_{42} + \quad x_{43} + \quad x_{44}) \end{array} \right\} \quad \ldots (12)$$

TABLE II

Augmented problem Associated with numerical problem

	Starting points of route 1	route 2	route 3	route 4	Augmented capacity $x_{i\,(4+1)}$	Existing capacity a_i
	90	126	54	72	630	
Depot 1	∞	∞	∞	∞	35	35
	1 30	1 5	1	1	−1	
	144	161	179	72	550	
Depot 2	∞	∞	∞	∞	50	30
	1	1 20	1 10	1	−1	
	108	36	144	108	610	
Depot (2 + 1)	∞	∞	∞	∞	20	0
	1	1 0	1	1	−1	
Artificial	M	M	M	M		
variable	∞	∞	∞	∞		
$x_{(2+1+1)j}$	1	1	1 30	1 35		
Buses required b_j	30	25	40	35		

TABLE III

Final tableau providing optimal basic feasible solution

	Starting points of route 1	route 2	route 3	route 4	Augmented capacity $x_{j\,(4+1)}$	Existing capacity a_i
	90	126	54	72	630	
Depot 1	∞ 26	∞ 45	∞	∞ 80	35	35
	1	1	1 40	1	—1 5	
	144	161	179	72	550	
Depot 2	∞	∞	∞ 45	∞	50	30
	1 30	1 5	1	1 35	−1 40	
	108	36	144	108	610	
Depot (2 + 1)	∞ 89	∞	∞ 135	∞ 160	20 −65	0
	1	1 20	1	1	−1 20	
Artificial	M	M	M	M		
Variable	∞ M-694	∞ M-711	∞ M-684	∞ M-622		
$x_{(2+1+1)j}$	1	1	1	1		
Buses required b_j	30	25	40	35		

Following the column-minima method after some modification as explained in section 3, an initial basic feasible solution of the augmented problem associated with the numerical problem is obtained and is shown in Table II. Skipping the intermediate steps, the final tableau providing the optimal basic feasible solution is shown in Table III. In Table III, values of all the relative cost coefficients corresponding to the nonbasic variables at zero level are nonnegative and at their upper bounds are nonpositive indicating that the basic feasible solution is optimal.

According to the optimal solution, capacities of depots 1 and 2 should be augmented to have 5 and 40 more buses respectively. A new depot should be constructed at site (2 + 1) to provide parking facility for 20 buses. All the 40 buses should be sent from depot 1 to the starting point of route 3. 30, 5 and 35 buses should be sent from the newly constructed depot (2 + 1) to the starting point of route 2. The minimum capital expenditure to be incurred in augmenting capacities of the depots plus the present value of the total cost associated with the dead-travelling of buses between the depots and the starting points of the routes over the planning horizon is 47875 thousand rupees.

5. Results and Discussions

Computer programmes for the solution procedure were developed and tested on ICL 2900 Computer. The solution procedure based on capacitated transportation method was found to be computationally more efficient than that based on branch and bound method of integer programming with respect to Central Processing Unit execution time and number of iterations.

6. Versatility and Applications of the Model

The model considered is quite versatile and has applications in areas other than urban bus transportation system. This can be seen if the terms 'depot' and 'starting point of route' used in the present work are stretched to include a supply point and a demand point respectively. With this generalization, the model considered can be employed to find optimal solutions to similar problems arising in areas of food grain movement from godowns to distribution centres and of crude petroleum supply from production sites to refineries.

7. Scope of Future Work

One of the areas for future work is the development of a technique superior to the existing branch and bound technique of integer programming and the one developed above for solving the problem considered in this work. This may be accomplished by exploiting the fact that the number of depots is quite small compared to the number routes in the problem considered in this work. The other area for future work is to solve the problem considered in in this work by introducing expressions for costs of augmentation of capacities of depots which could take care of an initial fixed expenditure associated with the augmentation of capacity of a depot.

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