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ON A CLASS OF NONLINEAR HIGHER ORDER DIFFERENTIAL EQUATIONS

B. G. PACHPATTE

Department of Mathematics and Statistics, Marathwada University, Aurangabad 431004 (Maharashtra)

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In this paper we study the existence, uniqueness, error estimations and continuous dependence of solutions of a class of nonlinear higher order differential equations with the given initial conditions. Our approach is based on converting the equations into equivalent integral equations and the application of the Wazewski's general method of successive approximations.

1. INTRODUCTION

Let $n \geqslant 2$ be an integer and let $p_i(t)$, $0 \leqslant i \leqslant n$, be positive continuous functions on $J = [t_0, a]$, $t_0 \geqslant 0$ and $a > 0$ is finite but can be arbitrarily large. We define the differential operators L_i, $0 \leqslant i \leqslant n$, by

$$L_0\, x(t) = \frac{x(t)}{p_0(t)},\quad L_i\, x(t) = \frac{1}{p_i(t)} \frac{d}{dt} L_{i-1}\, x(t),\ 1 \leqslant i \leqslant n.$$

In this paper we consider the nonlinear differential equation of the form

$$L_n\, x(t) = F(t, L_0\, x(t), L_1\, x(t), \ldots, L_{n-1}\, x(t)) \qquad \ldots(1)$$

with the initial conditions

$$L_{i-1}\, x(t_0) = C_{i-1},\ i = 1, \ldots, n \qquad \ldots(2)$$

where $F : J \times R^n \to R$ is a continuous function, C_{i-1} are given constants and R denotes the set of real numbers.

There are many papers written on the various special forms of (1) – (2) from different points of view, for example, see[1,3-5,11-16] and some of the references given there. In particular, Fink and Kusano[3] and Trench[11] have considered eqn. (1) and obtained conditions which imply that equation (1) has a solution x which behaves for large positive t like a given solution z of the unperturbed equation $L_n\, z = 0$. The main results in[3,11] are established by using the well known Schauder-Tychonoff fixed point theorem. The object of this paper is to establish some results on existence, uniqueness, error estimations of solutions of (1) — (2) and also continuous dependence of solutions on the right side of equation (1). Here we study the problem (1) — (2) by converting it into an equivalent integral equation and by using the general method of successive approximations based on the idea used by Wazewski[14] (see, also Kwapisz and Turo[7], Pachpatte[10]). Our formulation of the more general problem (1) — (2) is motivated in part by the studies of various types of equations by the authors[4,5,12,13,15,16] and the recent papers of Fink and Kusano[3] and Trench[11].

2. Statement of Results

We first convert the problem (1) — (2) into an equivalent integral equation. We say that x is a solution of (1) – (2) if $L_0 x, \ldots, L_n x$ exist and satisfy (1) – (2) on J. We shall be interested in solutions x which are continuous in J together with $L_0 x, \ldots, L_n x$. The set of all such solutions will be denoted by $C^* (J)$. If a function $x \in C^* (J)$ is a solution of (1) – (2), then for the function y continuous on J and defined by the formula $y (t) = L_n x (t)$, we have

$$\begin{aligned} L_{n-1} x (t) &= q_n (t) + I_n y (t) \\ L_{n-2} x (t) &= q_{n-1} (t) + I_{n-1} y (t) \\ &\vdots \\ L_1 x (t) &= q_2 (t) + I_2 y (t) \\ L_0 x (t) &= q_1 (t) + I_1 y (t) \end{aligned} \qquad \ldots(3)$$

where

$$\begin{aligned} q_n (t) &= C_{n-1} \\ q_{n-1} (t) &= C_{n-2} + C_{n-1} \int_{t_0}^{t} p_{n-1} (t_{n-1})\, dt_{n-1} \\ &\vdots \\ q_2 (t) &= C_1 + C_2 \int_{t_0}^{t} p_2 (t_2)\, dt_2 + \ldots \\ &\quad + C_{n-1} \int_{t_0}^{t} p_2 (t_2) \int_{t_0}^{t_2} \ldots \int_{t_0}^{t_{n-2}} p_{n-1} (t_{n-1})\, dt_{n-1} \ldots dt_2 \\ q_1 (t) &= C_0 + C_1 \int_{t_0}^{t} p_1 (t_1)\, dt_1 + \ldots \\ &\quad + C_{n-1} \int_{t_0}^{t} p_1 (t_1) \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_{n-2}} p_{n-1} (t_{n-1})\, dt_{n-1} \ldots dt_1 \end{aligned}$$

and

$$\begin{aligned} I_n y (t) &= \int_{t_0}^{t} p_n (s)\, y (s)\, ds \\ I_{n-1} y (t) &= \int_{t_0}^{t} p_{n-1} (t_{n-1}) \int_{t_0}^{t_{n-1}} p_n (s)\, y (s)\, ds\, dt_{n-1} \end{aligned}$$

(equation continued on p. 122)

$$\vdots$$

$$I_2\, y(t) = \int_{t_0}^{t} p_2(t_2) \int_{t_0}^{t_2} \cdots \int_{t_0}^{t_{n-2}} p_{n-1}(t_{n-1}) \int_{t_0}^{t_{n-1}} p_n(s)\, y(s)\, ds\, dt_{n-1} \ldots dt_2$$

$$I_1\, y(t) = \int_{t_0}^{t} p_1(t_1) \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-2}} p_{n-1}(t_{n-1}) \int_{t_0}^{t_{n-1}} p_n(s)\, y(s)\, ds\, dt_{n-1} \ldots dt_1.$$

Consequently the function y fulfils the equation

$$y(t) = F(t, q_1(t) + I_1\, y(t), q_2(t) + I_2\, y(t), \ldots, q_n(t) + I_n\, y(t). \qquad \ldots(4)$$

Conversely, if a function y, continuous on J, fulfils (4), then the function $x \in C^*(J)$ defined by (3) is a solution of (1) — (2). Thus the problem (1) — (2) is equivalent to the problem of solving integral equation (4). By substituting in the equation (4)

$$f(t, r_0, r_1, \ldots, r_{n-1})$$
$$= F(t, q_1(t) + r_0, q_2(t) + r_1, \ldots, q_n(t) + r_{n-1})$$

we get an integral equation of the form

$$y(t) = f(t, I_1\, y(t), I_2\, y(t), \ldots, I_n\, y(t)) = T\, y(t) \qquad \ldots(5)$$

with which we shall deal.

We make the following hypotheses used throughout this paper.

(A_1) Suppose that

(i) there exists a continuous function $g : J \times R_+^n \to R_+ = [0, \infty)$, nondecreasing with respect to the last n variables such that

$$g(t, 0, 0, \ldots, 0, 0) \equiv 0;$$

(ii) for $(t, r_0, r_1, \ldots, r_{n-1}), (t, \bar{r}_0, \bar{r}_1, \ldots, \bar{r}_{n-1}) \in J \times R^n$

$$|f(t, r_0, r_1, \ldots, r_{n-1}) - f(t, \bar{r}_0, \bar{r}_1, \ldots, \bar{r}_{n-1})|$$
$$\leqslant g(t, |r_0 - \bar{r}_0|, |r_1 - \bar{r}_1|, \ldots, |r_{n-1} - \bar{r}_{n-1}|).$$

(A_2) There exists a continuous function $\bar{u} : J \to R_+$ satisfying the inequality

$$Mu(t) + h(t) \leqslant u(t)$$

where

$$Mu(t) = g(t, I_1 u(t), I_2 u(t), \ldots, I_n u(t))$$

and

$$h(t) = \sup_{t_0 \leqslant \xi \leqslant t} |f(\xi, 0, 0, \ldots, 0)|. \qquad \ldots(6)$$

(A_3) In the class of functions satisfying the condition

$$0 \leqslant u(t) \leqslant \bar{u}(t), \; t \in J$$

the function $u, u(t) = 0$ for $t \in J$, is the only measurable solution of the equation

$$u(t) = Mu(t), \quad t \in J \qquad \dots(7)$$

where Mu is defined in (6).

In order to prove the existence of a solution of equation (5), we define the sequence $\{y_m\}$ by the relations

$$\left.\begin{aligned} y_0(t) &= 0 \\ y_{m+1}(t) &= Ty_m(t) \end{aligned}\right\} \qquad \dots(8)$$

for $t \in J$ and $m = 0, 1, 2, \dots$.

To prove the convergence of the sequence $\{y_m\}$ to the solution $\bar{y}$ of eqn. (5) we define the sequence $\{u_m\}$ by the relation

$$\left.\begin{aligned} u_0(t) &= \bar{u}(t) \\ u_{m+1}(t) &= Mu_m(t) \end{aligned}\right\} \qquad \dots(9)$$

for $t \in J$ and $m = 0, 1, 2, \dots$.

Now we shall state our results to be proved in this paper.

Theorem 1—Suppose that the hypotheses $(A_1) - (A_3)$ hold. Then there exists a continuous solution $\bar{y}(t)$, $t \in J$ of eqn. (5). The sequence $\{y_m\}$ defined by (8) converges uniformly to $\bar{y}$ in J and the following estimations

$$|\bar{y}(t) - y_m(t)| \leqslant u_m(t), \; t \in J, \; m = 0, 1, 2, \dots, \qquad \dots(10)$$

and

$$|\bar{y}(t)| \leqslant \bar{u}(t), \quad t \in J \qquad \dots(11)$$

hold. Moreover, the solution $\bar{y}$ of eqn. (5) is unique in the class of function satisfying the condition (11).

Our next result gives conditions under which eqn. (5) has at most one solution, these conditions do not guarantee the existence of a solution of eqn. (5).

Theorem 2—Let hypothesis (A_1) be fulfilled. If the function $r, r(t) = 0$, $t \in J$ is the only nonnegative, finite and measurable solution of the inequality

$$r(t) \leqslant Mr(t), \quad t \in J \qquad \dots(12)$$

then eqn. (5) has at most one solution on J.

In order to establish our next result which deals with the continuous dependence of solutions on the right side of eqn. (5), we consider the equation

$$z(t) = K(t, I_1 z(t), I_2 z(t), \dots, I_n z(t)) \qquad \dots(13)$$

where $K : J \times R^n \to R$ is a continuous function.

Theorem 3—Assume that the hypothesis (A_1) holds and

(i) $\bar{y}$ and $\bar{z}$ are solutions of eqns. (5) and (13) respectively;

(ii) the sequence $\{v_m(t)\}$, $t \in J$, defined by the relation

$$\left.\begin{aligned} v_0(t) &\geqslant |\bar{y}(t)| + |\bar{z}(t)| \\ v_{m+1}(t) &= Mv_m(t) + \bar{h}(t) \end{aligned}\right\} \quad \ldots(14)$$

for $t \in J$, $m = 0, 1, 2, \ldots$, where

$$\bar{h}(t) = |T\bar{z}(t) - \bar{z}(t)| \quad \ldots(15)$$

has a limit $\bar{v}(t)$ for $t \in J$. Then

$$|\bar{y}(t) - \bar{z}(t)| \leqslant \bar{v}(t), \; t \in J. \quad \ldots(16)$$

We note that, in the particular case where $p_i(t) \equiv 1$, $0 \leqslant i \leqslant n$, equations (1) and (2) reduces to

$$x^{(n)}(t) = F(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) \quad \ldots(17)$$

and

$$x^{(i-1)}(t_0) = C_{i-1}, \; i = 1, \ldots; n, \quad \ldots(18)$$

and consequently our results in Theorems 1-3 covers the study of equations (17)-(18). Here it is to be noted that the papers of Fink and Kusano[3] and Trench[11] are devoted to the study of asymptotic behaviour of solutions of (1).

3. Proofs of Theorems 1–3

Before we start the proofs of Theorems 1-3, we first prepare the following Lemma needed in our further discussion.

Lemma—If the condition (*i*) of hypothesis (A_1) and hypothesis $(A_2)-(A_3)$ are satisfied, then

$$0 \leqslant u_{m+1}(t) \leqslant u_m(t) \leqslant \bar{u}(t) \quad \ldots(19)$$

for $t \in J$, $m = 0, 1, 2, \ldots$, and

$$u_m \Rightarrow 0 \text{ for } m \to \infty$$

where the sign $\Rightarrow$ denotes the uniform convergence in J.

The relation (19) follows by induction. Since the sequence of continuous functions u_m is nonincreasing and bounded below, it is convergent to a certain measurable function ϕ such that $0 \leqslant \phi(t) \leqslant \bar{u}(t)$ for $t \in J$. By the Lebesgue theorem and the continuity of g it follows that the function ϕ satisfies equation (7) and by assumption

(A_3) we have $\phi(t) \equiv 0$, $t \in J$. The uniform convergence of $\{u_m\}$ in J follows from the Dini theorem. This completes the proof of Lemma.

In order to prove Theorem 1, first we prove that the sequence $\{y_m(t)\}$, $t \in J$ satisfies the condition

$$| y_m(t) | \leqslant \bar{u}(t), \ t \in J, \ m = 0, 1, 2, \ldots \qquad \ldots(20)$$

Obviously

$$| y_0(t) | = 0 \leqslant \bar{u}(t), \ t \in J.$$

Furthermore, if we suppose that the inequality (2) is true for $m \geqslant 0$, then by the definition of $y_m(t)$, $t \in J$ and by hypotheses (A_1) and (A_2), we have

$$\begin{aligned} | y_{m+1}(t) | &\leqslant M | y_m(t) | + h(t) \\ &\leqslant M\bar{u}(t) + h(t) \\ &\leqslant \bar{u}(t) \end{aligned}$$

for $t \in J$. The relation (2) follows by induction.

Next we prove that

$$| y_{m+q}(t) - y_m(t) | \leqslant u_m(t), \ t \in J, m, q = 0, 1, 2, \ldots \qquad \ldots(21)$$

By (20), we have

$$\begin{aligned} | y_q(t) - y_0(t) | &= | y_q(t) | \\ &\leqslant \bar{u}(t) = u_0(t) \end{aligned}$$

for $t \in J$, $q = 0, 1, 2, \ldots$, Suppose that (21) is true for $m,q \geqslant 0$, then

$$\begin{aligned} | y_{m+q+1}(t) - y_{m+1}(t) | &= | Ty_{m+q}(t) - Ty_m(t) | \\ &\leqslant M \ | y_{m+q}(t) - y_m(t) | \\ &\leqslant Mu_m(t) = u_{m+1}(t). \end{aligned}$$

Now we obtain (21) by induction. Because of Lemma $u_m(t) \Rightarrow 0$ in J, we have from (21) $y_m \Rightarrow \mathfrak{y}$ in J. The continuity of $\mathfrak{y}$ follows from the uniform convergence of the sequence $\{y_m\}$ and from the continuity of all functions y_m. If $q \to \infty$, then (21) gives estimation (10) and the estimation (11) is implied by (20). It is obvious that $\mathfrak{y}$ is a solution of (5).

To prove that the solution $\mathfrak{y}$ of (5) is unique, let us suppose that there exists another solution $\hat{y}$ of (5) such that $\mathfrak{y}(t) \not\equiv \hat{y}(t)$ and $| \hat{y}(t) | \leqslant \bar{u}(t)$ $t \in J$. By induction we get

$$| \hat{y}(t) - y_m(t) | \leqslant u_m(t), \ t \in J, \ m = 0, 1, 2, \ldots,$$

and hence it follows that $\mathfrak{y}(t) \equiv \hat{y}(t)$, $t \in J$. This contradiction proves the uniqueness of $\mathfrak{y}$ in the class of functions satisfying relation (11). This completes the proof of Theorem 1.

To prove Theorem 2, let us suppose that there exist two solutions $\mathfrak{y}$ and $\hat{y}$ of equation (5) in J, $\mathfrak{y}(t) \not\equiv \hat{y}(t)$, $t \in J$. Now, from hypothesis (A_1) we have for $t \in J$.

$$| \mathfrak{y}(t) - \hat{y}(t) | \leqslant M | \mathfrak{y}(t) - \hat{y}(t) | . \qquad ...(22)$$

Putting in (22), $r(t) = | \mathfrak{y}(t) - \hat{y}(t) |$, $t \in J$, we infer from (12) that $r(t) \equiv 0$ for $t \in J$, i. e. $\mathfrak{y}(t) \equiv \hat{y}(t)$, $t \in J$. This contradiction completes the proof of Theorem 2.

To prove Theorem 3, let

$$v(t) = | \mathfrak{y}(t) - \bar{z}(t) | , \; t \in J \qquad ...(23)$$

then we have

$$\begin{aligned} v(t) &\leqslant | T\mathfrak{y}(t) - T\bar{z}(t) | + | T\bar{z}(t) - \bar{z}(t) | \\ &\leqslant M | \mathfrak{y}(t) - \bar{z}(t) | + \bar{h}(t) \\ &= Mv(t) + \bar{h}(t). \end{aligned} \qquad ...(24)$$

From (23) and (14) we observe that

$$v(t) \leqslant | \mathfrak{y}(t) | + | \bar{z}(t) | \leqslant v_0(t), \; t \in J. \qquad ...(25)$$

Now by induction, we get

$$v(t) \leqslant v_m(t), \; t \in J, \; m = 0, 1, 2, ... \qquad ...(26)$$

Inequality (16) is implied by (26) as $m \to \infty$. This completes the proof of Theorem 3.

In concluding this paper, we note that the results obtained for eqns. (1) and (2) in Theorems 1-3 can be very easily extended for the integrodifferential equation of the form

$$L_n x(t) = F(t, L_0 x(t), L_1 x(t), ..., L_{n-1} x(t)),$$
$$\int_{t_0}^{t} H[t, s, L_0 x(s), L_1 x(s), ..., L_{n-1} x(s)] \, ds) \qquad ...(27)$$

with the given initial conditions (2), where $H : I^2 \times R^n \to R$, $F : I \times R^{n+1} \to R$ are continuous functions. Some results concering the existence, uniqueness and asymptotic behaviour of the solutions of the special versions of (27) – (2) when $p_i(t) \equiv 1$ have been obtained by the authors[8,9] by using different methods. The precise formulation of the results similar to that given in Theorems 1-3 for equations (27)—(2) are quite straight-forward and hence we do not discuss the details of these results.

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