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ON WAGNER SPACES OF W_p-SCALAR CURVATURE

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The principal purpose of the present paper is to introduce and study the notion of a Wagner space of W-perpendicular scalar curvature.

1. INTRODUCTION

Hashiguchi and Varga[1] examined Wagner spaces of W-scalar curvature as a generalization of Finsler spaces of scalar curvature and showed a result on Wagner spaces similar to that of Berwald spaces which was obtained independently by Numata[8] and Varga[11], Numata[9] gave a generalization of the above results.

On the other hand, Izumi and Yoshida[5] introduced and studied the notion of a space of perpendicular scalar curvature (abbreviated p-scalar curvature). Later they[6] gave correct form of the equation that characterizes a Finsler space of p-scalar curvature.

The main purpose of the present paper is to generalize the notion of a space of p-scalar curvature to a space of W-perpendicular scalar curvature (abbreviated W_p-scalar curvature) and to prove the main result :

Theorem— If a Finsler space is an s-Wagner space, and of W_p-scalar curvature, then the space is conformal to a Berwald space of p-scalar curvature.

Throughout the paper we shall use the terminology and notations of Matsumoto's monograph[7] with a few changes[4]. For example, $U_{(ij)}$ means interchange of indices i, j and subtraction; $\sigma_{(ijk)}$ does cyclic permutation of indices i, j, k and summation for the expression in the brackets behind it, and $A := B$ means that A is defined by B.

§1. WAGNER SPACES OF W_p-SCALAR CURVATURE

An n-dimensional Finsler space $F_n = (M_n, L)$ with a Finsler metric $L(x, y)$ is called a Wagner space of dimension n, if there exists a vector field s_i such that F^i_{jk} of the Wagner connection $W\Gamma$ with respect to s_i are functions of x^i only[1,2]. If $s_i(x)$ further satisfies $s_i(x) = \partial_i(s)$ $(\partial_i := \partial/\partial x^i)$, then F_n is called an s-Wagner space.

We denote by $(p.\,T)$ the projection of a tensor T on the indicatrix with respect to the Finsler metric[3,4] L. As the usual manner we raise or lower the indices by means of metric tensor g_{ij} and put $h_{ij} = g_{ij} - l_i l_j$ ($\dot\partial_i := \partial/\partial y^i$, $l_i := \dot\partial_i L$).

First we derive a Wagner analogy of the known identities[3,10]. That is, we shall obtain for an s-Wagner space :

$$R_{hljk} - R_{jkhl} = U_{(jk)} \left\{ C_{lkr} \; R^r_{hj} + C_{hjr} \; R^r_{lk} \right\} \quad \ldots(1.1)$$

$$\sigma_{(hjk)} \left\{ R_{hljk} - C_{hlr} \; R^r_{jk} \right\} = 0 \quad \ldots(1.2)$$

where R^r_{jk} are components of the (v) h-torsion tensor, R_{hljk} are components of the h-curvature tensor of $W\Gamma$. To establish it, we consider the expression[7] (10.18′)

$$R^l_{hjk} = K^l_{hjk} + C^l_{hr} \; R^r_{jk} \quad \ldots(1.3)$$

where

$$K^l_{hjk} = U_{(jk)} \left\{ \delta_k F^l_{hj} + F^r_{hj} F^l_{rk} \right\}$$

and $\delta_k := \partial_k - N^a_k \; \dot\partial_a$. For an s-Wagner space with respect to $W\Gamma$ noticing

$$s_l = \partial_l \; s, \; \delta_k \; F^l_{hj} = \partial_k F^l_{hj}$$

$$F^r_{hj} - F^r_{jh} = \delta^r_h \; s_j - \delta^r_j \; s_h$$

one can easily get $\sigma_{(hjk)} \left\{ K^l_{hjk} \right\} = 0$, which implies

$$\sigma_{(hjk)} \left\{ K_{hljk} \right\} = 0. \quad \ldots(1.4)$$

Then on account of (1.3) we obtain

$$\sigma_{(hjk)} \left\{ R^l_{hjk} - C^l_{hr} \, R^r_{jk} \right\} = 0$$

and thus (1.2). [(1.2) is also obtained in Hashiguchi and Varga[1] identity[7] (11.1′) of $W\Gamma$.]

Since $W\Gamma$ is h-metrical, we get $R_{hljk} = - R_{lhjk}$ and so

$$K_{hljk} + K_{lhjk} = -2C_{hlr} \, R^r_{jk} \, . \quad \ldots(1.5)$$

Now proceeding in similar way as in Rund[10] (§2, 2.25), we obtain (1.1). Here, on account of these identities (1.1) and (1.2), the results analogous to the Lemma 1.2[6] hold. Therefore we can apply the corresponding results in above lemma to prove Theorem 1.1.

We now consider a tangent vector $X = (X^i)$ of F_n at (x, y) and the $(v)h$-torsion tensor R^h_{jk} of $W\Gamma$. The quantity $K = K(x, y, X)$ defined by

$$R_{iok} X^i X^k = KL^2 h_{ik} X^i X^k \qquad ..(1.6)$$

at (x, y) is called the W-sectional curvature with respect to $W\Gamma$.

Definition[1]— A Finsler space F_n is said to be of W-scalar curvature K with respect to $W\Gamma$, if the W-sectional curvature K in (1.6) is a scalar field which does not depend on X.

For an s-Wagner space of W-scalar curvature K, $R_{iok} = KL^2 h_{ik}$ holds (Hashiguchi and Varga[1] and Numata[9]).

Now we are concerned with two independent vectors $X = (X^i)$ and $Y = (Y^i)$ in F_n. For the plane π $(p.X, p.Y)$ spanned by $p.X^i$ and $p.Y^i$, the W-perpendicular sectional curvature (abbreviated W_p-sectional curvature) $R := R(x, y, \pi(p.X, p.Y))$ with respect to $W\Gamma$ is defined by

$$R = \frac{R_{hljk}\,(p.X^h)\,(p.Y^i)\,(p.X^j)\,(p.Y^k)}{(g_{hj}\,g_{ik} - g_{hk}\,g_{ij})\,(p.X^h)\,(p.Y^i)\,(p.X^j)\,(p.Y^k)} \qquad ...(1.7)$$

where R_{hljk} is formed from the coefficients of the Wagner connection $W\Gamma$.
Thus we introduce :

Definition— A Finsler space F_n (≥ 3) is said to be of W_p-scalar curvature R with respect to $W\Gamma$, if the W_p-sectional curvature R in (1.7) is a scalar field which does not depend on X and Y.

Now we can prove :

Theorem 1.1— An s-Wagner space of W_p-scalar curvature is characterized by

$$p.R_{hljk} = U_{(jk)} \left\{ R\, h_{hj}\, h_{lk} + \tfrac{1}{2} \left(Q^r_{hj}\, C_{rlk} + Q^r_{lk}\, C_{rhj} \right) \right\} \qquad ...(1.8)$$

where

$$Q^r_{hj} := p.R^r_{hj}\,.$$

PROOF : Similar to the proof of the corresponding result in Izumi and Yoshida[6], and will be omitted.

From (1.7), it follows that the following condition

$$p.R_{hljk} = R\,(h_{hj}\,h_{lk} - h_{hk}\,h_{lj}) \qquad ...(1.9)$$

is a sufficient condition for a Wagner space to be of W_p-scalar curvature.

So, we may give the following :

Definition— A Wagner space F_n $(n > 2)$ characterized by (1.9) is called a Wagner space of W_{Rp}-scalar curvature.

§2. Proof of the Theorem

Let a Finsler space $F_n = (M_n, L)$ be a Wagner space with respect to a gradient vector field $s_l(x) = \partial_l s$, i.e. F_n is an s-Wagner space). Let $L^* = e^{-s(x)} L$ be a conformal transformation of Finsler metrics.

Now we consider the Finsler space $F^*_n = (M_n, L^*)$ and define

$$F^{*l}_{jk} = F^l_{jk} - \delta^l_j \, s_k$$

$$N^{*l}_k = N^l_k - y^l s_k,$$

$$C^{*l}_{jk} = C^l_{jk} \quad .$$

Since F^l_{jk} of the Wagner connection are function of x^i alone, so are the F^{*l}_{jk} of the Cartan connection $C\Gamma$ (Hashiguchi and Varga[1]). Hence F^*_n is a Berwald space. We denote, corresponding to quantities in F_n, quantities in F^*_n by an asterisk.

Next we assume that F_n is of W_p-scalar curvature, and so (1.8) holds.

Then since

$$g^*_{ij} = e^{-2s} g_{ij}, \; h^*_{ij} = e^{-2s} h_{ij}$$

$$R^{*r}_{hj} = R^r_{hj} \, , \; p.R^*_{hijk} = e^{-2s} p.R_{hljk}$$

and

$$C^*_{rik} = e^{-2s} C_{rik}$$

from (1.8) we have

$$p.R^*_{hijk} = U_{(jk)} \left\{ R \, e^{2s} h^*_{hj} h^*_{ik} + \tfrac{1}{2}\left(Q^{*r}_{hj} C_{rik} + Q^{*r}_{ik} C^*_{rhj} \right) \right\}$$

and consequently, by virtue of Theorem 1.1 of Izumi and Yoshida[6], the Berwald space F^*_n is of p-scalar curvature $R^* := Re^{2s}$, which completes the proof.

Similar to the above theorem we have the following result :

"If a Finsler space is an s-Wagner space, and of W_{R_p}-scalar curvature, then the space is conformal to a Berwald space of R_p-scalar curvature".

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