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GOLDIE THEOREM ANALOGUE FOR GOLDIE NEAR-RINGS

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Here we prove results on Goldie near-rings which are in some sense analogous to some results on Goldie theorem on Goldie rings. In some special types of near rings if the near rings of quotients are Goldie then the near rings are Goldie. An Abelian Goldie near ring K which is semiprime with respect to K subset and in which the non nilpotent elements are distributives, has a classical near ring Q of right quotient which is right Artinian and possesses no nonzero nilpotent right Q-subsets.

1. INTRODUCTION

In this paper we introduce the notion of a Goldie near-ring and prove some results in some sense analogous to Goldie theorems for Goldie rings.

A countable ordered family $\{A_1, A_2, \ldots\}$ of subsets of a (right) near-ring K is an independent family if for all $n \in N$, $A_i \cap (\sum_{k \neq i} A_k) = 0$, where $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n$.

A (right) near-ring K is a right Goldie near-ring if

(1) K satisfies the ascending chain condition (a.c.c.) for right annihilators and

(2) K has no infinite independent family of nonzero right K-subsets of K.

The ring $(Z, +, .)$ of integers and all finite near-rings are Goldie near-rings.

We note that a left annihilator of a subset of a (right) near-ring K is always a left ideal of it. Therefore a left Goldie near-ring can be defined as a right near-ring K with the following conditions :

(1) K satisfies the a.c.c. for left annihilators.

(2) K has no direct sum of an infinite numbers of left ideals.

In our present study we shall confine ourselves to right Goldie near-rings only.

A K-subset P of a near-ring K is 'prime' if for any two K-subsets A and B of K, $AB \subseteq K \Rightarrow A \subseteq K$ or $B \subseteq K$, we shall see the existence of such a K-subsets in a Goldie near-ring K.

Here we shall call a near-ring 'prime' if 0 is a prime K-subset, i.e., if for any two K-subsets A and B of K, $AB = 0 \Rightarrow A = 0$ or $B = 0$.

A right (left) K-subset A of K is nil if each element of A is nilpotent.

A right (left) K-subset A of K is 'nilpotent' if there exists $n \in Z^+$ such that for any $a_1, \ldots, a_n \in A$, $a_1 \ldots, a_n = 0$. There exist examples of such K-subsets of a near-ring K. (Clay[2], 2, 13).

We shall call a near-ring K 'semiprime' if K has no nonzero nilpotent K-subset of K.

We can prove, as in case of Goldie rings (a ring with the a. c. c. on annihilator ideals and having no nonzero infinite direct sum of ideals) that the direct sum of two Goldie near-rings is again Goldie. We can also prove the following interesting results: In a semiprime Goldie near-ring the collection $\mathcal{B}$ of minimal prime K-subsets of K is finite and $\bigcap_{P \in \mathcal{B}} P = 0$. Another interesting result on semiprime Goldie near-ring is that if for some $P \in \mathcal{B}$, $P = r(A)$ the right annihilator of a distributive K-subset A, then the quotient near-ring $\bar{K} = K/P$ is a prime Goldie near-ring.

We know that : If K is a regular near-ring whose idempotents are central then $(K, +)$ is Abelian[3]. And also there exists Abelian near-ring whose idempotents are the only non-nilpotent elements[2]. We now consider Goldie near-rings which is additively commutative (Abelian near-ring) and whose non nilpotent elements are distributive with distributively closed right essential K-susets. In this paper we establish that if the classical near-ring Q of right quotients of an Abelian near-ring K in which non nilpotent elements are distributive as above and all the idempotent elements of Q commute with the regular elements of K is such that Q is a Goldie near-ring with d.c.c. on right Q-subset and possesses no nonzero nil right Q-subset, then K is a Goldie near-ring and has no nonzero nilpotent right K-subset. Next we prove that a semiprime Abelian Goldie near-ring in which non-nilpotent elements are distributive as above has a classical near-ring Q of right quotients which is right Artinian and possesses no nonzero nilpotent right Q-subsets.

2. Preliminaries

Here we assume that a near-ring K contains unity and for every $a \in K$, $a.0. = 0$.

We write $Q(K)$ to denote the complete near-ring of right quotients of K (Barua[1]).

A near-ring K satisfies the (right) Ore condition with respect to a subset S of it, if given $(a, r) \in K \times S$, there exists a common right multiple $ar' = ra'$ such that $(a', r') \in K \times S$. If K satisfies the (right) Ore condition with respect to S, the set of nonzero divisors (regular elements) of K, then the subset $Q = \{ar^{-1} \in Q(K) \mid (a,r) \in K \times S\}$ is a subnear-ring of $Q(K)$. Q is the classical near-ring of right quotients of K.

Let K be a subnear-ring of a near-ring L, A any subset of L. Then

$$r_L(A) = \{x \in L \mid ax = 0, \text{ for all } a \in A\}$$

$$r_K(A) = \{x \in K \mid ax = 0, \text{ for all } a \in A\}.$$

Clearly $r_L(A)$ is a right L-subset of L and $r_K(A)$ is a right K-subset of K.

For any $A, B \subseteq K$, $A + B = \{a + b \mid a \in A, b \in B\}$. A countable ordered family $\{A_1, A_2, \ldots\ldots\}$ of subsets of a near-ring K is an 'independent family' if for all $n \in N$, $A_i \cap (\sum_{k \neq i} A_K) = 0$, where $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n$.

Let K be a near-ring and A, B are right K-subsets of K such that $A \subseteq B$. Then A is right K-essential in B if for any nonzero right K-subset C contained in B we have $A \cap C \neq 0$. A subset A of K is 'right essential K-subset' if it is right K-essential in K.

A near-ring K is '(right) non singular' if for any right essential K-subset A of K and $z \in K$, $zA = 0$ implies $z = 0$.

A 'right annihilator ideal' $I = r(s)$ $(= r_K(s))$ of K is a right annihilator K-subset of K such that I is a right ideal of K.

A near-ring K is right Artinian if it satisfies the d.c.c. on right ideals.

Since a semiprime near-ring K can not have any nonzero nilpotent right (left) K-subset and since in a Goldie near-ring it is possible to choose a maximal right annihilator of the type $r(a)$ where a is a nonzero element in a right (left) K-subset A of K, a semiprime Goldie near-ring K has no nonzero nil right (left) K-subset.

We now give some results (Lemmas) for use in the proofs of the main results in §3. The following are easy to prove.

Lemma 2.1.1— If $s_1; \ldots, s_n \in S$, the set of regular elements of K, then there exist $k_1, \ldots k_n \in K$, $s \in S$ such that $s_i^{-1} = k_i s^{-1}$, $i = 1, \ldots, n$.

For any subset A of K we write

$$QA = \{\sum_{\text{fin.}} q_i a_i \mid q_i \in Q, a_i \in A\}$$

$$AS^{-1} = \left\{\sum_{\text{fin.}} a_i s_i^{-1} \mid a_i \in A, s_i \in S\right\}.$$

If J is a right K-subset of K then by Lemma 2.1.1 we get

$$JS^{-1} = \{js^{-1} \mid j \in J, s \in S\}.$$

Lemma 2.1.2— JS^{-1} is a right Q-subset of Q.

Lemma 2.1.3— If K is additively commutative, then so is Q.

Lemma 2.1.4— If K is an Abelian near-ring and $A \subseteq K$ then

$$QAQ = \{ \sum_{\text{fin.}} x_i \, a_i \, y_i \mid x_i, y_i \in Q, a_i \in A\}$$

is a right Q-subgroup of Q. (The result follows from the right distributivity in Q).

Lemma 2.1.5— If $T \subseteq K$ and $J = r(T)$, then $JS^{-1} = r_Q(T)$ and $JS^{-1} \cap K = J$.

Lemma 2.2.1— Let the non nilpotent elements in the near-ring K be distributive. If K satisfies the d.c.c. on right K-subsets of K then every non nil right K-subset I of K contains a nonzero idempotent element.

PROOF : Let $F = \{J \subseteq I \mid J$ is a non-nil right K-subset of $K\}$.

Since $I \in F$, $F \neq \phi$ and thus F contains a minimal element, say I_1. I_1 being non nil I_1^2 is also non nil and $I_1^2 \subseteq I_1$. So, by minimality of I_1, $I_1^2 = I_1$. Now consider the family of all non nil right K-subset J of K with the property that $J I_1 \neq 0$ and $J \subseteq I_1$. This family is also non empty, for I_1 is an element in this family. Thus it contains a minimal element, say J_1. Thus $J_1 I_1 \neq 0$ and J_1 is non nil. Let $u \in J_1$ be a non nilpotent element. So $U I_1 = 0$ and $u I_1 \subseteq J_1$. By minimality of J_1, $uI_1 = J_1$. Therefore for some $a \in I_1$, $u a = u$, which gives $u = ua^n$ for all $n \in Z^+$. And u being nonzero, a is non nilpotent and so it is distributive. Let $A(u) = \{r \in I_1 \mid ur = 0\}$. $A(u)$ is a right K-subset of K. And $u \notin A(u)$, $u \in I_1$ gives $A(u) \subset I_1$. So by minimality of I_1, $A(u)$ is nil. Since u is non nilpotent, it is distributive. Therefore $u(a^2 - a) = ua^2 - ua = ua - ua = 0$. Thus $a^2 - a \in A(u)$. Hence $a^2 - a$ is nilpotent and let for $n \in Z^+$, $(a^2 - a)^n = 0$. Since a is distributive we get on expanding, $a^n = a^{n+1} g(a)$, where $g(x)$ is a polynomial in x with coefficients $+1$ or -1. Since a^n is also distributive, we therefore get $a^n g(a) = g(a) a^n$. Now $a^n = a^{n+1} g(a) = a(a^n g(a)) = a(a^{n+1}(g(a))^2) = a^{n+2}(g(a))^2$. Continuing this process finally we get, $a^n = a^{2n}(g(a))^2$. We write $e = a^n g(a)^n$. Since $a^n \in I$. And it can be seen that $e \neq 0$ and $e^2 = e$.

Lemma 2.2.2— If K in Lemma 2.2.1 is Abelian and K has no nonzero nil right K-subset, then for every nonzero right K-subgroup I of K there exists an idempotent element $e \in K$ such that $I = eK$.

PROOF : By Lemma 2.2.1, I contains an idempotent element, say e^1. Now $A(e^1) = \{x \in I \mid e^1 x = 0\}$ is a right K-subset of K. The family of all such sets $A(e^1)$, where e^1 is an idempotent in I, possesses a minimal element, say $A(e)$. If $A(e) \neq 0$ then it contains an idempotent, say e_1. So $ee_1 = 0$. Write $e_2 = e + e_1 - e_1 e$. Since K is Abelian and e, e_1 are distributive we get $e_2^2 = e_2$. And I, being a right K-subgroup, $e_2 \in I$. Moreover, $e_2 x = 0$ implies $ex = 0$ for $ee_2 = e$. Therefore $A(e_2) \subseteq A(e)$. Again $e_1 \in A(e)$ but $e_1 \notin A(e_2)$, for $e_2 e_1 = e_1 \neq 0$. Thus $A(e_2) \subset A(e)$. And minimality of $A(e)$ therefore implies that $A(e) = 0$. So for any $x \in I$, $ex \neq 0$ implies $x = 0$. But for any $x \in I$, $e(x - ex) = 0$. Therefore $x = ex$. Hence $I = eI \subseteq eK \subseteq I$ which gives $I = eK$.

It is easy to see the following :

Lemma 2.3.1— Let A, B, C be right K-subsets of a near-ring K such that $A \subseteq B \subseteq C \subseteq K$ and A is right K-essential in B; B is right K-essential in C. Then A is right K-essential in C.

Next we prove :

Lemma 2.3.2— Let N be a right K-subset of K and M be such a right K-subset of K that M is right K-essential in N.

If $a \in N$, $a \neq 0$, then there is a right essential K-subset L of K such that $aL \neq 0$ and $aL \subseteq M$.

PROOF : Write $L = \{k \in K \mid a\, k \in M\}$. Clearly L is a right K-subset of K and $a\, L \subseteq M$. Since N is a right K-subset of K, $a\, K \subseteq N$, $aK \neq 0$ and $aK \cap M \neq 0$ (Since $1 \in K$ and M is right K-essential in M). Therefore there is some $k \in K$ such that $ak \in M$, $ak \neq 0$. So $aL \neq 0$. Now let I be a nonzero right K-subset of K. If $aI = 0$, then $I \subseteq L$ and hence $I \cap L \neq 0$. And if $aI \neq 0$, $aI \subseteq N$ and M is right K-essential in N give that $aI \cap M \neq 0$. So for some $x \in I$, $ax \neq 0$, $ax \in M$. This implies that $x \in I \cap L$. Since $x \neq 0$, $I \cap L \neq 0$. Thus L is a right essential K-subset of K.

Taking $N = K$ and $M = A$, a right essential K-subset of K we get the

Corollary 2.3.3— For any $a \in A$, $a^{-1} A = \{x \in K \mid a\, x \in A\}$ is a right essential K-subset of K.

Lemma 2.3.4— Let A, B be right annihilator K-subsets of K with $A \subseteq B$ and A be right K-essential in B. If K is right nonsingular then $A = B$.

PROOF : Let $b \in B$, $b \neq 0$. Since $A \subseteq B$ and A is right K-essential in B, by Lemma 2.3.2 there exists a right essential K-subset L of K such that $bL \subseteq A$, $bL \neq 0$. Thus $l(A)\, bL = 0$ and K being right nonsingular we get $l(A)\, b = 0$ which gives $b \in r(l(A))$. Since A is of the type $r(S)$ for some S K, $r(l(A)) = A$. Hence $b \in A$ whence $B \subseteq A$. Thus $A = B$.

Lemma 2.3.5— If A and B are two right K-subsets of a right nonsingular near-ring K such that $A \subseteq B$ and A is right K-essential in B, then for any $x \in K$ the subset xA is right K-essential in xB.

PROOF : Let $C \subseteq xB$, $C \neq (0)$ be a right K-subset of K and let $c = xb$ be a nonzero element of C. Then for $b \in B$ there is one right essential K-subset L of K such that $bL \neq 0$, $bL \subseteq A$ (Lemma 2.3.2). Since K is right nonsingular, we therefore get, $xbL \neq 0$, for otherwise we shall get $xb = 0$. Now $xA \cap C \supseteq xA \cap xbK \supseteq xbL \neq 0$, which implies that xA is right K-essential in xB.

Lemma 2.4.1— Let K be a Goldie near-ring whose non-nilpotent elements are distributive. If $x \in K$ is such that $r(x) = 0$, then xK is a right essential K-subset of K.

PROOF : Since $r(x) = 0$, x is non nilpotent and therefore it is distributive. Let M be a right K-subset of K such that $M \cap xK = 0$. Now for a fix $s \in Z^+$ and for $t \leqslant s$, let

$$\alpha \in (\sum_{n \neq t} x^n M) \cap x^t M, \ (x^0 = 1, n = 0, 1, \ldots, s).$$

Then

$$\alpha = \sum_{n \neq t} x^n m_n = x^t m_t, \ m_t, m_n \in M.$$

i.e.,

$$m_0 = x^t m_t - x^s m_s - \ldots - x m_1$$

$$= x(x^{t-1} m_{t-1} - x^{s-1} m_{s-1} \ldots - m_1) \text{ (since } x \text{ is distributive).}$$

Thus, $m_0 \in M \cap xK$ which gives $m_0 = 0$. It follows that

$$x^{t-1} m_{t-1} - x^{s-1} m_{s-1} - \ldots m_1 = 0, \text{ for } r(x) = 0$$

Similarly we get $m_1 = m_2 = \ldots = m_t = 0$. Therefore $(\sum_{n=t} x^n M) \cap x^t M = 0$ for all $s \in Z^+$ and $t \leqslant s$. Thus the family $\{M, xM, x^2 M, \ldots\ldots\}$ is an independent family. K being Goldie, there exists $u \in Z^+$ such that $x^{u+1} M = 0$. So for any $m \in M$, $x^{u+1} m = 0$ which gives $m = 0$ (Since $r(x) = 0$). Therefore $M = 0$. Thus $M \cap xK = 0$ implies $M = 0$. Hence xK is a right essential K-subset of K.

Lemma 2.4.2— The right singular K-subgroup of a Goldie near-ring K is nilpotent. (Follows from Proposition 4.3.6 in Barua[1]). Therefore

Lemma 2.4.3— A semiprime Goldie near-ring is right non singular.

Lemma 2.4.4— If K is a semiprime Goldie near-ring where non-nilpotent elements are distributive then every distributively closed right essential K-subset of K contains a regular element.

PROOF : Let I be any distributively closed right essential K-subset of K. We first show that I contains an element a such that $r(a) = 0$.

Since K is semiprime and $I \neq 0$, it is not nil. Let us choose a non nilpotent element $a_1 \in I$ with $r(a_1)$ as large as possible (it is possible for K is Goldie). Suppose $r(a_1) \neq 0$. Then $r(a_1) \cap I \neq 0$. As above choose a non nilpotent element $a_2 \in r(a_1) \cap I$ with $r(a_2)$ as large as possible. And suppose $r(a_1 + a_2) \neq 0$. Now $r(a_1 + a_2) \cap I \neq 0$ and it is not nil. Choose a non nilpotent element $a_3 \in r(a_1 + a_2) \cap I$ with $r(a_3)$ as large as possible. We see that $r(a_1) \cap r(a_2) = r(a_1 + a_2)$. Now we note that $r(a_1) \cap r(a_2) \subseteq r(a_1 + a_2)$. First we show $a_1 K \cap a_2 K = 0$. Let $z = a_1 x = a_2 y$

belong to the intersection. Since a_1^2, a_2^2 are non nilpotent, by maximality of $r(a_1)$, $r(a_2)$ we get $r(a_1^2) = r(a_1)$ and $r(a_2^2) = r(a_2)$. Therefore $a_1^2 x = a_1 a_2 y = 0$, for $a_2 \in r(a_1)$. Then $x \in r(a_1^2)$, i.e. $x \in r(a_1)$. Thus $z = 0$. Hence $a_1 K \cap a_2 K = 0$. Now if $x \in r(a_1 + a_2)$, then $a_1 x = a_2(-x)$, for a_2 is distributive. Since $a_1 K \cap a_2 K = 0$, We get $x \in r(a_1) \cap r(a_2)$. Hence $r(a_1 + a_2) = r(a_1) \cap r(a_2)$. As above we see that $\{a_1 K, a_2 K\, a_3 K\}$ is an independent family and $r(a_1 + a_2 + a_3) = r(a_1) \cap r(a_2) \cap r(a_3)$ We continue this process. Because of Goldie condition we get a non left zero divisor $C = a_1 + \ldots\ldots + a_n$ i.e. $r(c) = 0$. We claim that $l(c) = 0$. Let $xc = 0$ for some $x \in K$. Then $xcK = 0$. By Lemma 2.4.1, cK is a right essential K-subset of K. And K being semiprime, it is right non singular. So $xcK = 0$ implies $x = 0$. Hence $l(c) = 0$ Thus c is a regular element of K.

Lemma 2.4.5— Let K be a semiprime Abelian Goldie near-ring such that its non-nilpotent elements are distributive. Then K-satisfies the d.c.c. for right annihilator ideals.

Proof : Let A, B be two right annihilator ideals. Then $A = r(T)$ and $B = r(S)$, $S, T \subset K$. If $A \subset B$ and A is not right K-essential in B, then by Lemma 2.3.4. there exists a right K-subset $P\ (\subset B)$ such that $A \cap P = 0$. K being semiprime Goldie, P is not nil. Let p be a non nilpotent element in P and so it is distributive and pK is a right ideal of K, for K is Abelian. Now $A \cap pK \subseteq A \cap P = 0$ i.e. $A \cap pK = 0$. If possible let $B \supset A \supset C$ be a strictly deceending chain of right annihilator ideals where $B = r(T)$, $A = r(S)$, $C = r(U)$. As above there is a right K subset $Q \subset A$ such that $C \cap Q = 0$. And as in case of P, there is a distributive element $q \in Q$ such $C \cap qK = 0$. We claim that $\{pK, qK, C\}$ is an independent family. For $A \cap pK = 0$, $C \cap qK = 0$, $pK \subseteq BC$, $qK \subseteq A$. Hence $pK \cap (qK + C) \subseteq pK \cap (A + C) \subseteq pK \cap A = 0$. If $qk_1 = pk_2 + c \in qK \cap (pK + C)$, then $pk_2 = qk_1 + (-c) \in (qK + C) \cap pK = 0$. Thus $qk_1 = c \in C \cap qK = 0$. Similarly $C \cap (pK + qK) = 0$. So an infinite strictly descending chain of right annihilator ideals gives an infinite independent family of right K-subsets of K (for K is Abelian), which contradicts the Goldie character of K. Hence K satisfies the d.c.c. for right annihilator ideals.

3. Main Results

We now give the main results.

Theorem 3.1— If the near-ring Q is Goldie then so is K.

Proof : Let $J_1 \subset J_2 \subset \ldots$, where $J_i = r(T_i)$, $T_i \subseteq K$ be a strictly ascending chain of right annihilator K-subsets of K. Then by Lemma 2.1.5.

$J_1 S^{-1} \subseteq J_2 S^{-1} \subseteq \ldots\ldots$ is an ascending chain of right annihilator Q-subsets of Q. The near ring Q being Goldie, $J_m S^{-1} = J_{m+1} S^{-1} = \ldots\ldots$, for some $m \in Z^+$. And by Lemma 2.1.5 we therefore get $J_m = J_{m+1} = \ldots\ldots$ Therefore K cannot have any infinite strictly ascending chain of right annihilator K-subsets. Next if $\{J_1, \ldots\ldots, J_t\}$ is an independent family of right K-subsets of K, we claim that the family $\{J_1 S^{-1}, \ldots, J_t S^{-1}\}$ of right Q-subsets of Q is independent. If for some $m, l \leqslant m \leqslant t$, $J_m S^{-1} \cap (\sum_{n \neq m} J_n S^{-1}) \neq 0$, then we get a nonzero element $j_m\, s_m^{-1} = \sum_{n \neq m} j_n\, s_n^{-1}$ in the intersection. By Lemma 2.1.1, we get $k_1, \ldots, k_t \in K$, $s \in S$ such that $s_i^{-1} = k_i s^{-1}$, $l \leqslant i \leqslant t$ And because of the right distributivity in Q,

$$j_m\, k_m\, s^{-1} = \sum_{n \neq m} j_n\, k_n\, s^{-1} = (\sum_{n \neq m} j_n\, k_n)\, s^{-1} \text{ which gives}$$

$$j_m\, k_m = \sum_{n \neq m} j_n\, k_n. \text{ Thus } J_n \cap (\sum_{n \neq m} J_n) = 0.$$

This contradicts that the family $\{J_1, \ldots, J_t\}$ is independent. Therefore K can not have an infinite independent family of right K-subsets.

Thus K is Goldie.

Theorem 3.2— Let K be an Abelian near-ring in which non nilpotent elements are distributive and the regular elements of K commute with the idempotent elements of its classical near-ring Q of right quotients which satisfies the d.c.c. for right Q-subsets and which possesses no nonzero nil right Q-subset. Then K is a Goldie near-ring with no nonzero nilpotent right K-subset.

Proof : By Theorem 3.1 K is Goldie.

Now suppose N is a nilpotent right K subset of K such that $N^2 = 0$. By Lemma 2.1.4. QNQ is a right Q-subgroup of Q. So by Lemma 2.2.2. $QNQ = eQ$ for some nonzero idempotent $e \in Q$. Therefore $e = \sum_{\text{fin.}} x_i\, n_i\, y_i$, $x_i, y_i \in Q$, $n_i \in N$ and each $y_i = k_i\ s_i^{-1}$, $k_i \in K$, $s_i \in S$. Using Lemma 2.1.1. We get $u_i \in K$, $s \in S$ such that each $s_i = u_i\, s^{-1}$, $i = 1, \ldots\ldots, t$. Therefore $e = (\sum_{\text{fin.}} x_i\, n_i\, k_i\, u_i)\, s^{-1}$. And this gives $es = \sum_{\text{fin.}} x_i\, n_i\ k_i\, u_i \in QN$ (since each $n_i\, k_i\, u_i \in N$). So $esN \subseteq QN^2$, i.e., $esN = 0$. Since e commutes with s, $seN = 0$ which gives $eN = 0$. And $N \subseteq QNQ = eQ$ gives that for $n \in N$, $n = eq$, $q \in Q$. Hence $n = en \in eN$. Thus $N \subseteq eN$, i.e., $N = 0$.

Theorem 3.3— A semiprime Abelian Goldie near-ring K in which non nilpotent elements are distributive with distributively closed right essential K-subsets, has a classical near-ring Q of right quotients which has no nilpotent right Q-subsets.

Proof : Choose $a, b \in K$, a is regular in K. K being Goldie, by Lemma 2.4.1, a K is right essential in K. Then, by Corollary 2.3.3., the set $\lambda = \{k \in K \mid bk \in aK\}$ is right essential in K Therefore by Lemma 2.4.4, λ contins a regular element, say a^1. Thus $ba^1 = ak^1$, for some $k^1 \in K$. Thus the right Ore condition with respect to the set S of regular elements of K is satisfied in K. So by Lemma 5.4.4, (Barua[1]) K has a classical near-ring of right quotients of K, say Q.

Next let J be a right Q-subset of Q such that $J^2 = 0$. Now $J \cap K$ is a right K-subset of K. Because of Lemma 2.1.1, $(J \cap K) Q = \{xc^{-1} \mid x \in J K, c \in S\}$. Now for any $x \in J$, $x = ks^{-1}$, $k \in K$, $s \in S$. So $xs = k \in J K$. Thus $x = (xs) s^{-1} \in (J K) Q$. Conversely if $ys^{-1} \in (J \cap K) Q$, $y \in J \cap K$, $s \in S$, then $ys^{-1} \in J$. Hence $J = (J \cap K) Q$. Again $J^2 = 0$ gives $(J \cap K^2) (\subseteq J^2) = 0$. Therefore $J \cap K$ is a nil-potent right K-subset of K and K is semiprime. Hence $J \cap K = 0$. Thus it follows from what we have showed above that $J = 0$.

Theorem 3.4— A semiprime Abelian Goldie near-ring K in which non-nilpotent elements are distributive has a classical near-ring Q of right quotients which is right Artinian.

Proof : First we show that if A, B are two right ideals of Q with $B \subseteq A$ and $B \cap K$ is right K-essential in $A \cap K$ then $A = B$. Let $x \in A \cap K$. By Lemma 2.3.2, there is a right essential K-subset L of K such that $xL \subseteq B \cap K$. And by Lemma 2.4.4, L contains a regular element, say $c \in L$. Thus $x c \in B \cap K$ and $x = (xc)c^{-1} \in ((B \cap K) Q =) B$. Therefore $A \cap K \subseteq B$. Hence $A = (A \cap K) Q \subseteq BQ \subseteq B$. Thus $A = B$.

So if $B \subset A$, then $B \cap K$ is not right K-essential in $A \cap K$ which implies that there is a nonzero right K-subset X of K contained in $A \cap K$ such that $X \cap (B \cap K) = 0$. Since X cannot be nil, it contains a distributive element, say x. Again $xK \cap (B \cap K) = 0$ and xK is a right ideal of K. Further we have $xK \subseteq X \subseteq A \cap K$. If $A \supset B \supset C \supset D$, where C, D are right ideals of Q, then in like manner we get right K-subsets, Y, Z of K contained in $B \cap K$ and $C \cap K$ respectively such that $yK \subseteq Y \subseteq B \cap K$, $zK \subseteq Z \subseteq C \cap K$, $yK \cap (C \cap K) = 0$ and $zK \cap (D \cap K) = 0$. We show that $\{xK, yK, zK\}$ is an independent family. Let $xk_1 = yk_2 + zk_3 \in xK \cap (yK + zK)$. Then we get $yk_2 + zk_3 \in xK \cap (B \cap K)$ (since $B \cap K$ is a right ideal of K). Thus $xk_1 = 0$, i.e. $xK \cap (yK + zK) = 0$. Similarly $yK \cap (xK + zK) = 0 = zK \cap (xK + yK)$. Therefore the family $\{xK, xK, zK\}$ is independent. Thus an infinite strictly descending chain of right ideals of Q gives an infinite independent family of right K-subsets of K which is in contradiction with the Goldie character of K. Hence Q must be right Artinian.

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