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THERMO-ELASTIC WAVES FROM SUDDENLY PUNCHED HOLE IN STRETCHED ELASTIC PLATE

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Coupled thermo-elastic wave problem of infinitely extended elastic plate of thickness d (subject to initially a state of axially symmetric hydrostatic tension) a flate nose cylindrical projectile of radius C travelling with velocity V strikes the plate and begins to punch out a hole of radius equal to its own, has been studied under cretain assumption. The integral transform technique is used. The expression for stresses and temperature are derived. Long time and short time solution are presented.

INTRODUCTION

The elastic deformation due to time dependent surface traction (from Duhamel—Neuman analysis) can be determined without reference to its thermal state. This procedure is adopted in the solution of dynamical problem in the classical theory of elasticity. The elastic constants which appear in the equation of motion are defined under adiabatic condition and since elastic waves being non-dispersive produce no increase in the entropy of the solid, no inconsistency seem to arise. This description of elastic wave propagation is physically over simplified. In fact a change in volume must cause the temperature as well as stresses. When a longitudinal wave passes through a solid the elements are successively compressed and dialated. These phenomena are accompanied respectively by heating and cooling. Since the thermal conductivity of the solid is non-zero and the disturbance has finite frequency, the source of energy will be converted into heat energy during the first half of an oscillation will not be recovered during the dialation phase.

With the foregoing remarks in mind this paper is prepared. A formal solution of coupled thermo-elastic problem of infinitely extended elastic plate of thickness d subject to initially a state of axially symmetric hydrostatic tension i.e., $\sigma_r = \sigma_\theta = \Delta$, a flate nose cylindrical projectile of radius C travelling with velocity V strikes the plate and begins to punch out a hole of radius equal to its own has been obtained. The problem is solved under certain assumptions :

(a) The plastic flow due to punching is very localized to the neighbourhood of the punch sections.

(experiment also support for $V \geqslant 2000$ fps).

(b) The punching begins instaneously at $t = 0$ over the whole punched section, based on a small value of plate thickness d and large value of impact speed V.

(c) The punching action takes place at velocity $V/2$, the particle's velocity in the compressional wave that develop in both projectile and plate on impact i.e., the plate material below the projectile is removed as a plug at $V/2$. The corresponding punching time is therefore $2d/V = l$, based on large ratio of diameter of projectile to plate thickness.

A few problems are solved on thermo-mechanical coupling effect[1] Kumar uncoupled problem Miklowitz[2]. The result of Miklowitz[2] are obtained as particular case.

2. Mathematical Formulation and Solution of the Problem

The origin of the cylindrical co-ordinates (r, θ, Z) is taken on the axis of the cylindrical hole. For radially symmetric case the only non-zero displacement component is $u\,(r, t)$ in the radial direction for plane stress problem.

The thermo-elastic equation of motion in the absence of body forces can be written[3],

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\,(\sigma_{rr} - \sigma_{\theta\theta}) = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots(1)$$

and conduction equation

$$\rho C_v K\,\nabla^2 T = \rho C_v \frac{\partial T}{\partial t} + m\,T_0\,\frac{\partial e}{\partial t} \qquad \ldots(2)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} : \; m = (3\lambda + 2\mu)\,\alpha_t.$$

The strain-displacement relation and stresses are :

$$e_{rr} = \frac{\partial u}{\partial r} \; : \; e_{\theta\theta} = \frac{u}{r}$$

$$\sigma_{rr} = \lambda e + 2\mu \frac{\partial u}{\partial r} - (3\lambda + 2\mu)\,\alpha_t\,(T - T_0)$$

$$\sigma_{\theta\theta} = \lambda e + 2\mu \frac{u}{r} - (3\lambda + 2\mu)\,\alpha_t\,(T - T_0)$$

$$e = e_{rr} + e_{\theta\theta} = \frac{\partial u}{\partial r} + \frac{u}{r} \qquad \ldots(3)$$

where λ, μ are Lame's constants; α_t is the coefficient of linear expansion; $(T - T_0)$ the deviation from the equilibrium temperature T_0; ρ the density; C_v the specific heat at constant cubical dialation; and K the conductivity of the material.

The boundary conditions of the problem are:

$$\left.\begin{aligned} \sigma_{rr} &= 0;\ t < 0 \\ &= -\frac{\Delta t}{l} : 0 < t < l \\ &= -\Delta : t > l \end{aligned}\right\} \qquad r = a \qquad l = \frac{2d}{V}. \qquad \ldots(4)$$

From the relations (1), (2), (3), we can write equation of motion as,

$$\frac{\lambda + 2\mu}{\rho}\frac{\partial^2 u}{\partial r^2} + \frac{\lambda + 2\mu}{\rho}\frac{\partial}{\partial r}\frac{u}{r} - \frac{(3\lambda + 2\mu)}{\rho}\alpha_t \frac{\partial}{\partial r}(T - T_0) = \frac{\partial^2 u}{\partial t^2} \qquad \ldots(5)$$

and Conduction equation :

$$\nabla^2 T = \frac{\rho C_v}{K}\frac{\partial T}{\partial t} + \frac{mT_0}{K}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right). \qquad \ldots(6)$$

By Introducing the following non-dimensional quantities

$$R = \frac{C_1}{\kappa} r : \tau = \frac{C_1}{\kappa} t \; U = \frac{\rho C_1^2 u}{m T_0 \kappa}$$

$$T_1 = \frac{T - T_0}{T_0} \quad C_1^2 = \frac{(\lambda + 2\mu)}{\rho} \qquad \ldots(7)$$

$\kappa = \dfrac{K}{\rho C_v}$ is the coefficient of thermal diffusivity (7). Equation (5) and (6) reduces to

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{U}{R^2} - \frac{\partial^2 U}{\partial \tau^2} = \frac{\partial T_1}{\partial R} \qquad \ldots(8)$$

$$\frac{\partial^2 T_1}{\partial R^2} + \frac{1}{R}\frac{\partial T_1}{\partial R} = \frac{\partial T_1}{\partial \tau} + \delta \frac{\partial}{\partial \tau}\left(\frac{\partial U}{\partial R} + \frac{U}{R}\right) \qquad \ldots(9)$$

where

$$\delta = \frac{\{(3\lambda + 2\mu)\,\alpha_t\}^2}{\rho C_v (\lambda + 2\mu)} \times T_0 \text{ is a coupling factor.}$$

Equation (9) and (8) can be written as

$$\left.\begin{aligned} &\text{(a)}\ \frac{\partial}{\partial R}\left(\frac{\partial U}{\partial R} + \frac{U}{R} - T_1\right) = \frac{\partial^2 U}{\partial \tau^2} \\ &\text{(b)}\ \left(\frac{\partial}{\partial R} + \frac{1}{R}\right)\left(\frac{\partial T_1}{\partial R} - \delta\frac{\partial U}{\partial \tau}\right) = \frac{\partial T_1}{\partial \tau}. \end{aligned}\right\} \qquad \ldots(10)$$

We define transformation

$$\left.\begin{aligned}&\text{(a) } \bar{f}(R, p) = \int_0^\infty f(R, \tau)\, e^{-p\tau}\, d\tau\\&\text{and its inversion is}\\&\text{(b) } f(R, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f(R\ P)}\, e^{p\tau}\, dp.\end{aligned}\right\} \quad \ldots(11)$$

Taking transform of eqns. (10)

$$\left.\begin{aligned}&\text{(a) } \frac{\partial}{\partial R}\left(\frac{\partial \bar{U}}{\partial R} + \frac{\bar{U}}{R} - \bar{T}_1\right) = p^2\, \bar{U}\\&\text{(b) } \left(\frac{\partial}{\partial R}\frac{1}{R}\right)\left(\frac{\partial \bar{T}_1}{\partial R} - \delta\, p\, \bar{U}\right) = p\, \bar{T}_1\end{aligned}\right\} \quad \ldots(12)$$

where 'Bar' stands for transformed function and p is transform variable with the assumption of

$$D = \frac{\partial}{\partial R} : D^1 = \left(\frac{\partial}{\partial R} + \frac{1}{R}\right) : \begin{aligned} D_1 &= DD^1\\ D_2 &= D^1\, D.\end{aligned}$$

Equations (12) can be written as

$$\left.\begin{aligned}&\text{(a) } (DD^1 - p^2)\, \bar{U} = D\bar{T}_1\\&\text{(b) } (D^1\, D - p)\, \bar{T}_1 = \delta\, pD^1\, \bar{U}.\end{aligned}\right\} \quad \ldots(13)$$

Operating D_1, D_2 on (a) and (b) respectively, we get

$$\left[D_1^2 - \left(m_1^2 + m_2^2 \right) D_1 + m_1^2\ m_2^2 \right] \bar{U} = 0 \quad \ldots(14)$$

$$\left[D_2^2 - \left(m_1^2 + m_2^2 \right) D_2 + m_1^2\ m_2^2 \right] \bar{T}_1 = 0 \quad \ldots(15)$$

where m_1^2 and m_2^2 are the roots of the equation

$$m^4 - m^2\,[p^2 + (1 + \delta)\, p] + p^3 = 0. \quad \ldots(16)$$

Equation (14) is equivalent to

$$\left.\begin{aligned}&\text{(a) } \left(D_1 - m_1^2\right)\ \bar{U}_1 = 0\\&\text{(b) } \left(D_1 - m_2^2\right)\ \bar{U}_2 = 0;\quad \bar{U} = \bar{U}_1 + \bar{U}_2\end{aligned}\right\} \quad \ldots(17)$$

and eqn. (15) is equivalent to

$$\left.\begin{array}{l}\text{(a)} \left(D_2 - m_1^2\right) \bar{T}_{11} = 0 \\ \text{(b)} \left(D_2 - m_2^2\right) \bar{T}_{12} = 0; \qquad \bar{T}_1 = \bar{T}_{11} + \bar{T}_{12}.\end{array}\right\} \quad \ldots(18)$$

The general solutions consistent with the boundary conditions are

$$\left.\begin{array}{l}\bar{U} = \bar{U}_1 + \bar{U}_2 = AK_1(m_1 R) + BK_1(m_2 R) \\ \bar{T}_1 = \bar{T}_{11} + \bar{T}_{12} = EK_0(m_1 R) + FK_0(m_2 R)\end{array}\right\} \quad \ldots(19)$$

where A, B, E, F are arbitrary constants and $K_0(Z)$, $K_1(Z)$ are modified Bessel function of second kind of order zero and one respectively.

Substituting the values of $\bar{U}\,\bar{T}_1$ from (19) in eqns. 13 we get the following relations in constants

$$E = \frac{(p^2 - m_1^2)}{m_1} A, \qquad F = \frac{(p^2 - m_2^2)}{m_2} B \quad \ldots(20)$$

and thus

$$\bar{U} = AK_1(m_1 R) + B\,K_1(m_2 R)$$

$$\bar{T}_1 = \frac{(p^2 - m_1^2)}{m_1} .\, A\,K_0(m_1 R) + \frac{p^2 - m_2^2}{m_2} B\,K_0(m_2 R).$$

As the displacements are known, we can compute the components of stress and temperature.

$$\begin{array}{l}\overline{\sigma_{RR}} = A\,M_1(m_1 R) + B\,M_2(m_2 R) \\ \bar{T}_1 = A\,N_1(m_1 R) + B\,N_2(m_2 R)\end{array} \quad \ldots(21)$$

where

$$\beta^2 = (\mu/\lambda) + 2\mu$$

$$M_1(m_1 R) = \frac{p^2}{m_1} K_0(m_1 R) + \beta^2 K_1(m_1 R)$$

$$M_2(m_2 R) = \frac{p^2}{m_2} K_0(m_2 R) + \beta^2 K_1(m_2 R)$$

$$N_1(m_1 R) = \frac{(p^2 - m_1^2)}{m_1} K_0(m_1 R)$$

$$N_2(m_2 R) = \frac{(p^2 - m_2^2)}{m_2} K_0(m_2 R). \quad \ldots(22)$$

The rest constanss A, B can be determined by using the boundary conditions (4) after taking their transformation, and equating (21), we get

$$A\,M_1\,(m_1\,b) + B\,M_2\,(m_2\,b) = \frac{\Delta}{l}\frac{1}{p^2}\,(e^{-lp} - 1)$$

$$A\,N_1\,(m_1\,b) + B\,N_2\,(m_2\,b) = 0. \qquad \ldots(23)$$

Solving above two equations for A and B, we get

$$A = \frac{\Delta\,(1 - e^{-lp})\,N_2\,(m_2\,b)}{lp^2\,[M_2\,(m_2\,b)\,N_1\,(m_1\,b) - M_1\,(m_1\,b)\,N_2\,(m_2\,b)]}$$

$$B = \frac{-\,\Delta\,(1 - e^{-lp})\,N_1\,(m_1\,b)}{lp^2\,[M_2\,(m_2\,b)\,N_1\,(m_1\,b) - M_1\,(m_1\,b)\,N_2\,(m_2\,b)]} \qquad \ldots(24)$$

where $b = \frac{c_1}{\kappa}\,a$

After substituting the values of A and B in (19) we get displacement and temperature in image space

$$\bar{U} = \frac{\Delta\,(1 - e^{-lp})}{l\,p^2\,F\,(p)}\,[N_2\,(m_2\,b)\,K_1\,(m_1\,R) - N_1\,(m_1\,b\,K_1\,(m_2\,R)]$$

$$\bar{T}_1 = \frac{\Delta\,(1 - e^{-lp})}{l\,p^2\,F\,(p)}\left[\frac{(p^2 - m_1^2\,)}{m_1}\,N_2\,(m_2\,b)\,K_0\,(m_1\,R)\right.$$

$$\left. - \frac{(p^2 - m_2^2\,)}{m_2}\,N_1\,(m_2\,b)\,K_0\,(m_2\,R)\right]$$

where

$$F\,(p) = M_2\,N_1 - M_1\,N_2. \qquad \ldots(25)$$

Long Time Solution

The long time solution can be obtained by expanding the roots m_1^2, m_2^2 of (16) for small value of p. The expansion is done in Taylor's series[4]

$$m_1 = (1 + \delta\tfrac{1}{2})\,\sqrt{p} + O\,(p)^{3/2}$$

$$m_2 = (1 + \delta)^{-1/2}\,p + O\,(p)^2 \qquad \ldots(26)$$

Neglecting higher power of p and denoting as

$$m_1 = l_1\,\sqrt{p} \quad \text{where } l_1 = (1 + \delta)^{-1/2}$$

$$m_2 = l_2\,p \qquad l_2 = (1 + \delta)^{-1/2}. \qquad \ldots(27)$$

By replacing the values of m_1 and m_2 from (27) in (25), we get displacement and temperature in image space for long time solutions,

$$\bar{U} = \frac{\Delta (1 - e^{-tp})}{l p^2 F(p)} [N_2 (l_2 p b) K_1 (l_1 V\sqrt{p} R) - N_1 (l_1 p b) K_1 (l_2 p R)]$$

$$\bar{T}_1 = \frac{\Delta (1 - e^{tp})}{l p^2 F(p)} \left[\frac{(p^2 - l_1^2 p)}{l_1 V p} N_2 (l_2 p b) K_0 (l_1 V^- p R) - \frac{(p^2 - l_2^2 p^2)}{l_2 p} N_1 (l_1 \sqrt{p} b) K_0 (l_2 p R) \right] \quad ...(28)$$

and similarly the stress components $\overline{\sigma_{RR}} (R, p)$, $\overline{\sigma_{\theta\theta}} (R, p)$ and temperature $\bar{T}_1 (R, p)$ can be evaluated in image space.

Short Time solution

The short time solution can also be obtained by expanding the roots m_1^2, m_2^2 of (16) for large value of p. The expansion is done by Laurent series[4] :

$$m_1^2 = p^2 + \delta p + \delta + \frac{\delta (1 - \delta)}{p} + \frac{\delta (1 - 3\delta + \delta^2)}{p^2} + O (p^{-3})$$

$$m_2^2 = p - \delta - \frac{\delta (1 - \delta)}{p} - \frac{\delta (1 - 3\delta + \delta^2)}{p^2} + O (p^{-3})$$

or

$$\left. \begin{aligned} m_1 &= p + \frac{\delta}{2} + \frac{\delta (4 - \delta)}{8p} + O (p^{-2}) \\ m_2 &= \sqrt{p} - \frac{\delta}{2\sqrt{p}} - O (p^{-3/2}) \end{aligned} \right\} . \quad ...(29)$$

Neglecting higher powers of p, $m_1 = l_1$; $m_2 = l_2$ where

$$l_1 = p + \frac{\delta}{2} + \frac{\delta (4 - \delta)}{8p}$$

$$l_2 = \sqrt{p} - \frac{\delta}{2\sqrt{p}}. \quad ...(30)$$

By substituting the values of m_1 and m_2 from (30) we get corresponding displacements and temperature for short time solutions in image space.

$$\bar{U} = \frac{\Delta (1 - e^{-pt})}{l p^2 F (p)} [N_2 (l_2 b) K_1 (l_1 R) - N_1 (l_1 b) K_1 (l_2 R)] \quad ...(31)$$

$$\bar{T}_1 = \frac{\Delta(1 - e^{-lp})}{l p^2 F(p)} \left[\frac{(p^2 - l^2)}{l_1} N_2(l_2 b) K_0(l_1 R) - \frac{(p^2 - l_2^2)}{l_2} N_1(l_1 b) K_0(l_2 R) \right] \quad \ldots(32)$$

and similarly stress components, $\overline{\sigma_{RR}}, \overline{\sigma_{\theta\theta}}$ can be evaluated in image space.

The stress components $\sigma_{RR}(R, \tau)$ and $T_1(R_1 \tau)$ can be obtained by taking the inversion of the transform.

$$\sigma_{RR}(R_1 \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{\sigma_{RR}(R\, p)} e^{p\tau}\, dp.$$

Similarly all other components and temperature can be evaluated. This formally completes the problem.

3. Discussion of Results

We have discussed the above problem under thermo-mechanical coupling effects.

In eqns. (17) we have noticed that there appears one more displacement compared to uncoupled problem[2] which is responsible for thermal waves in the medium. As a matter of fact consideration of Thermo-mechanical coupling is desirable for all elasto-dynamic problem at least for high speed impact problems otherwise we neglect a good amount of energy.

The thermo-mechanical coupling is done by a coupling factor δ and if $\delta = 0$ the temperature from (19) will vanish.

If $\delta = 0$ the displacement function agrees with Miklowitz[2] in image space and thus con-sequently stresses.

If $\delta = 0$, $t = 0$ the problem reduces to uncoupled stationary problem.

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