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NOTE ON MINMAX PRINCIPLE FOR HEAT CONVECTION EQUATION

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A variational principle for the energy equation with viscous dissipation subject to Dirichlet condition is obtained.

1. INTRODUCTION

Barret *et al.*[1] have given a minmax principle for a non-self adjoint system and applied it to Navier-Stokes equations.

Here, we give a minmax principle for the heat convection equation with viscous dissipation when the surface temperature is prescribed. For simplicity, we mention the situations where we may obtain the saddle functional.

2. MINMAX PRINCIPLE FOR ENERGY EQUATION

The differential equation for the steady distribution of temperature in a viscous incompressible liquid is given by

$$\kappa \nabla^2 T = \rho c_v \bar{V} . \nabla T - \frac{1}{J} \Phi \qquad \ldots(2.1)$$

where κ is the Thermal conductivity of fluid,

C_v the heat capacity per unit volume of fluid,

J the mechanical equivalent of heat,

Φ the viscous dissipation function, and

$\bar{V}$ the fully developed laminar velocity of fluid.

The boundary condition considered is

$$T = T_s \text{ on the boundary } S. \qquad \ldots(2.2)$$

Introduce the functional

$$J(T_1, T_2) = \int_V [\kappa ((\nabla T_1)^2 - (\nabla T_2)^2) - T_2 \bar{V} . \nabla T_1 + T_1 \bar{V} . \nabla T_2 - 2\alpha (T_1 - T_2) \Phi] \, dV \qquad \ldots(2.3)$$

Subject to

$$T_1 - T_2 = T_s \text{ on } S \qquad ...(2.4)$$

where

$$\kappa = (\kappa/\rho\, C_v),\ \alpha = (1/\rho\, C_v\, J).$$

Equating the first variation in J to zero, we get

$$\kappa\, \nabla^2 T_1 = \bar{V}.\, \nabla T_2 - \alpha\, \Phi \qquad ...(2.5)$$

$$\kappa\, \nabla^2 T_2 = \bar{V}.\, \nabla T_1 - \alpha\, \Phi. \qquad ...(2.6)$$

It is easily seen that

$$T_1 \equiv T_2.$$

Then (2.5) and (2.6) reduce to (2.1) and finding the stationary point of the functoinal (2.3) is equivalent to solving the original problem.

If T_2 is fixed, the second variation in J is

$$\delta^2 J = \int_V \kappa\, (\nabla\xi)^2\, dV \qquad ...(2.7)$$

where ξ is an admissible function. Keeping T_1 fixed, we have

$$\delta^2 J = - \int_V \kappa\, (\nabla\, \eta)^2\, dV \qquad (2.8)$$

where η is an admissible function. From (2.7) and (2.8) we notice that if T_2 is fixed, J is minimum with respect to T_1 and if T_1 is fixed, J is maximum with respect to T_2. Therefore, J represents a saddle functional and the solution is obtained from

$$\underset{T_2}{\text{Max}}\ \underset{T_1}{\text{Min}}\ J\,(T_1, T_2) \qquad ...(2.9)$$

where the trial functions satisfy the appropirate boundary conditions. The following solution algorithm is suggested :

(1) Choose T_2 to satisfy $T_2 = T_s$ on S.

(2) Solve

$$\kappa\, \nabla^2 T_1 = \bar{V}.\, \nabla T_2 - \alpha\, \Phi,\ T_1 = T_s \text{ on } S.$$

At the end of this step, the corresponding value of J is

$$J\,(T_1\,(T_2), T_2) = - \int_V \kappa\, (\nabla T_1 - \nabla T_2)^2\, dV \qquad ...(2.10)$$

providing a direct estimate of the error.

(3) Maximize J with respeet to T_2 and thus obtain the unknown parameters in T_2.

Note that for the time-dependent temperature equation

$$\kappa \nabla^2 T = \frac{\partial T}{\partial t} + \bar{V}.\nabla T - \alpha \Phi \qquad ...(2.11)$$

with the initial and boundary conditions

$$\left.\begin{array}{l} T = 0 \text{ at } t = 0 \\ T = T_s \text{ on } S, t > 0 \end{array}\right\} \qquad ...(2.12)$$

the saddle functional may be found to be

$$J(T_1, T_2) = \int_t \int_V [\kappa ((\nabla T_1)^2 - (\nabla T_2)^2) - T_2 \bar{V}.\nabla T_1 + T_1 \bar{V}.\nabla T_2 - 2\alpha (T_1 - T_2)\Phi - T_2 \frac{\partial T_1}{\partial t} + T_1 \frac{\partial T_2}{\partial t}\Big] dV\, dt. \qquad ...(2.13)$$

3. Applications

(A) *Generalized Couette Flow with Suction and Injection*

Consider the steady two-dimensional flow of a viscous incompressible fluid between two parallel flat plates. The equation for the temperature distribution with viscous dissipation is[2]

$$\frac{d^2 T^*}{d\eta^2} = PR \frac{dT^*}{d\eta} - PE \left(\frac{du^*}{d\eta}\right)^2 \qquad ...(3.1)$$

subject to the conditions

$$T^*(0) = 0, T^*(1) = 1. \qquad ...(3.2)$$

The saddle functional is found to be

$$J(T_1, T_2) = \int_0^1 \Big(\Big(\frac{dT_1}{d\eta}\Big)^2 - \Big(\frac{dT_2}{d\eta}\Big)^2 - PR\, T_2 \frac{dT_1}{d\eta} + PRT_1 \frac{dT_2}{d\eta} - 2PE(T_1 - T_2)\Big(\frac{du^*}{d\eta}\Big)^2\Big) d\eta. \qquad ...(3.3)$$

The algorithm similar to section 2 is used to obtain the solution and the estimate of the error is given by

$$J(T_1(T_2), T_2) = -\int_0^1 \Big(\frac{dT_1}{d\eta} - \frac{dT_2}{d\eta}\Big)^2 d\eta. \qquad ...(3.4)$$

(B) *Parabolic Flow Between Two Semi-infinite Parallel Plates*

Consider the parabolic flow between two semi-infinite parallel plates $y = 0$, $y = 2h$ maintained at a constant temperature T_0, the temperature of the incident fluid being zero. Neglecting the axial heat conduction and viscous dissipation, the governing equation for the fluid temperature is[3]

$$\kappa \frac{\partial^2 T}{\partial y^2} = U(y) \frac{\partial T}{\partial x} \qquad \text{...(3.5)}$$

where $U(y)$ represents the fully developed laminar velocity in channel. The saddle functional is seen to be

$$J(T_1, T_2) = \int_0^\infty \int_0^{2h} \left[\kappa \left(\left(\frac{\partial T_1}{\partial y} \right)^2 - \left(\frac{\partial T_2}{\partial y} \right)^2 \right) - U(y)\, T_2 \frac{\partial T_1}{\partial x} \right.$$

$$\left. + \, U(y)\, T_1 \frac{\partial T_2}{\partial x} \right] dy\, dx \qquad \text{...(3.6)}$$

and the corresponding error estimate is

$$J(T_1(T_2), (T_2)) = - \int_0^\infty \int_0^{2h} \kappa \left(\frac{\partial T_1}{\partial y} - \frac{\partial T_2}{\partial y} \right)^2 dy\, dx. \qquad \text{...(3.7)}$$

We notice that for the slug flow between parallel plates, the saddle functional is represented by (3.6), where

$$U(y) = U_0 \text{ (constant)}.$$

(C) *Magnetohydrodynamic Pipe Flow*

Consider the steady flow of a conducting liquid along a pipe under the influence of a transverse magnetic field. If S represents a cross-section of the pipe in the (x, y) plane, the fluid velocity w and induced magnetic field h satisfy[4]

$$\nabla^2 w + M \frac{\partial h}{\partial y} = -1 \text{ in } S \qquad \text{...(3.8)}$$

$$\nabla^2 h + M \frac{\partial w}{\partial y} = 0 \text{ in } S \qquad \text{...(3.9)}$$

with

$$w = h = 0 \text{ on } \partial S$$

for stationary insulated walls, M being the Hartmann number.

Equations (3.8) and (3.9) may be uncoupled by addition and subtraction to give

$$\nabla^2 P + M \frac{\partial P}{\partial y} = -1 \text{ in } S \qquad \text{...(3.10)}$$

$$\nabla^2 Q - M \frac{\partial Q}{\partial y} = -1 \text{ in } S \qquad ...(3.11)$$

where

$$P = w + h, \qquad Q = w - h$$

and

$$P = Q = 0 \text{ on } \partial S. \qquad ...(3.12)$$

The saddle functional for (3.10) is found to be

$$J(P_1, P_2) = \iint_S \Big[(\nabla P_1)^2 - (\nabla P_2)^2 + M P_2 \frac{\partial P_1}{\partial y} - M P_1 \frac{\partial P_2}{\partial y} - 2(P_1 - P_2) \Big] dx\, dy \qquad ...(3.13)$$

and that the corresponding saddle functional for (3.11) namely, $J(Q_1 Q_2)$, is obtained by replacing M by $(-M)$ P_1 by Q_1 and P_2 by Q_2 in (3.13). These principles are then used to obtain the approximate fluid velocity w and induced magnetic field h from

$$w = \tfrac{1}{2}(P + Q) \text{ and } h = \tfrac{1}{2}(P - Q).$$

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