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A FINSLERIAN EXTENSION OF THE GRAVITATIONAL FIELD—II

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Some structural considerations are further made on the previously introduced Finslerian metrical structure $g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y)$. In particular, some interesting features underlying the field equations for Finslerian gravitational field based on this metric are investigated.

1. INTRODUCTION

As has been mentioned in a previous paper[1], there appear two fields around each point x of the Finslerian gravitational field : One is the external (x)-field spanned by points $\{x\}$, which is nothing else than the Einstein's gravitational field[2], and the other is the internal (y)-field spanned by vectors $\{y\}$, which is regarded as the so-called internal space attached to x. The former is dominated by the four-dimensional Riemann (R_4) metric $\gamma_{\lambda\kappa}(x)$ $(\kappa, \lambda = 1, 2, 3, 4)$, while the latter is assumed, in general, to be governed by the four-dimensional Riemann (R_4) metric $h_{ij}(y)$ $(i, j = 1, 2, 3, 4)$.

From the vector bundle-like standpoint[4], the (y)-field may be regarded as a fibre at the point x of the base (x)-field and the total space of this vector bundle may be considered a unified field between the (x) and (y)-fields. The Finslerian gravitational field, therefore, may be likened to this unified field, presenting an aspect of eight-dimensional Riemannian structure (R_8) dominated by the R_8-metric $G_{AB}(X^A)$ $(X^A = (x^\kappa, y^i); A = (\kappa, i) = 1, 2, 3, \ldots, 8)$ (Miron[3] and Miron and Anastasiei[4]). This R_8 structure has been reduced, in the previous paper[1], to a four-dimensional Finslerian structure (F_4) based on the Finsler metric $g_{\lambda\kappa}(x, y)$ (2.9) by means of a dimension reduction-process. These situations will be considered in more detail in the following.

2. ON THE FINSLERIAN STRUCTURE

In the previous paper (see section 3 of Ikeda[1]), we have taken account of inherent law internal vector y $(= y^i)$ in the form

$$y^i = K^i_j(x)\, y^j \qquad \ldots(2.1)$$

where K^i_j represents the rotation matrix. From (2.1), one typical intrinsic parallelism (i.e., connection) of y such as

$$\delta y^i = dy^i + K^i_{j\mu}\, y^j\, dx^\mu \;(\equiv dy^i + N^i_\mu\, dx^\mu) = 0 \qquad \ldots(2.2)$$

has been proposed, where $K^i_{j\mu} \equiv -\dfrac{\partial K^i_j}{\partial x^\mu}$ and $dy^i \equiv \bar{y}^i - K^i_j(0)\, y^j$. The quantity N^i_μ in (2.2) plays the role of nonlinear connection in the theory of Finsler spaces[3,5] and also serves, in our case[1], as the mapping operator of the internal (y)-field on the external (x)-field (see below).

In our Finslerian field, which is regarded as the unified field mentioned above, the connection relations are prescribed by[3,4] [see (3.4) of Ikeda[1]]

$$DV^\kappa = dV^\kappa + F^\kappa_{\lambda\mu} V^\lambda dx^\mu + C^\kappa_{\lambda k} V^\lambda \delta y^k$$

$$DV^i = dV^i + F^i_{j\mu} V^j dx^\mu + C^i_{jk} V^j \delta y^k \qquad \ldots(2.3)$$

where

$$(F^\kappa_{\lambda\mu},\, F^i_{j\mu},\, C^\kappa_{\lambda k},\, C^i_{jk}) \qquad \ldots(2.4)$$

denote the coefficients of connection. From (2.3), the following covariant derivatives are introduced :

$$\left.\begin{aligned} V^\kappa{}_{|\mu} &= \frac{\delta V^\kappa}{\delta x^\mu} + F^\kappa_{\lambda\mu} V^\lambda, \quad V^\kappa|_k = \frac{\partial V^\kappa}{\partial y^k} + C^\kappa_{\lambda k} V^\lambda \\ V^i{}_{|\mu} &= \frac{\delta V^i}{\delta x^\mu} + F^i_{j\mu} V^j, \quad V^i|_k = \frac{\partial V^i}{\partial y^k} + C^i_{jk} V^j. \end{aligned}\right\} \qquad \ldots(2.5)$$

Therefore, the base and dual base (i.e., the adapted frames) are set by

$$\left.\begin{aligned} \frac{\partial}{\partial \zeta^A} &\equiv \left(\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^i_\mu \frac{\partial}{\partial y^i},\; \frac{\partial}{\partial y^i}\right) \\ d\zeta^A &\equiv (dx^\kappa,\; \delta y^i = dy^i + N^i_\mu\, dx^\mu). \end{aligned}\right\} \qquad \ldots(2.6)$$

From (2.6), the so-called decomposition factors are defined by

$$\left.\begin{aligned} A^\kappa_A &= (\delta^\kappa_\lambda,\, 0),\; A^A_\lambda = (\delta^\kappa_\lambda,\, -N^i_\lambda), \\ B^i_A &= (N^i_\lambda,\, \delta^i_j),\; B^A_i = (0,\, \delta^j_i) \end{aligned}\right\} \qquad \ldots(2.7)$$

by which the unified metric of the unified field G_{AB} is decomposed to, under the assumption that $G_{\lambda\kappa} = \gamma_{\lambda\kappa}(x)$, $G_{ij} = h_{ij}(y)$ and $G_{\lambda i} = G_{i\lambda} = 0$,

$$\left.\begin{aligned} g_{\lambda\kappa} &= A^A_\lambda \; A^B_\kappa \; G_{AB} = \gamma_{\lambda\kappa}(x) + N^i_\lambda \; N^j_\kappa \; h_{ij}(y) \\ g_{\lambda i} &= A^A_\lambda \; B^B_i \; G_{AB} = - \; N^j_\lambda \; h_{ji}(y) \\ g_{ij} &= B^A_i \; B^B_j \; G_{AB} = h_{ij}(y). \end{aligned}\right\} \qquad \ldots(2.8)$$

At this stage, as has been considered in Ikeda[1], if the (y)-field, is compactified, then only the F_4-metric $g_{\lambda\kappa}(x, y)$ $(2.8)_1$ appears above the surface. This compactification is likened geometrically to the mapping process of the (y)-field on the (x)-field, where the nonlinear connection N^i_λ plays the role of mapping operator. (In the previous paper[1], one new operator $e^i_\lambda(x)$, instead of $N^i_\lambda(x, y)$, has been introduced as the mapping operator). By this mapping process, those quantities such as δy^i, $F^i_{j\mu}$, $C^\kappa_{\lambda k}$, C^i_{jk}, etc. are brought to the external (x)-field and the unified field presents an aspect of F_4-structure based on

$$g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y);$$

$$h_{\lambda\kappa}(x, y) \equiv N^i_\lambda \; N^j_\kappa \; h_{ij}(y). \qquad \ldots(2.9)$$

In the following, we shall pay our attention to some structural features underlying the field equations for this Finslerian gravitational field from the vector bundle-like standpoint based on (2.4).

3. On the Field Equations

Now, in our theory, the (y)-field itself is dominated by R_4-metric $h_{ij}(y)$, as mentioned in (2.8), so that its field equation may be written as

$$M_{ij}(y) - \tfrac{1}{2} M \, h_{ij}(y) = \mu_{ij}(y) \qquad \ldots(3.1)$$

where M_{ij} denotes the Ricci-tensor derived from the Riemannian curvature tensor of the (y)-field, M the scalar curvature and μ_{ij} means the energy-momentum tensor for this case. Then, (3.1) must be mapped on the external (x)-field governed by R_4-metric $\gamma_{\lambda\kappa}(x)$ by means of the mapping process mentioned above. As the result, therefore, the field equation of the unified is given by

$$R_{\lambda\kappa} - \tfrac{1}{2} R \, g_{\lambda\kappa} = \tau_{\lambda\kappa} \qquad \ldots(3.2)$$

where $R_{\lambda\kappa}$ is the Ricci-tensor derived from the third curvature tensor $R^\kappa_{\nu\lambda\mu}$ formed with $F^\kappa_{\lambda\mu}$, $R = R_{\kappa\lambda} \, g^{\kappa\lambda}$ and $\tau_{\lambda\kappa}$ denotes the energy-momentum tensor. In our, case, since $g_{\lambda\kappa}$ is given by (2.9), $R_{\lambda\kappa}$ has such a special form as

$$R_{\lambda\kappa}(F) = K_{\lambda\kappa}(\{\}) + M_{\lambda\kappa}(\Delta) \qquad \text{...(3.3)}$$

where $K_{\lambda\kappa}$ represents the Riemannian Ricci-tensor, which is formed with the Christoffel three-index symbol $\left\{ {\kappa \atop \lambda\mu} \right\}$ with respect to $\gamma_{\lambda\kappa}(x)$, and $M_{\lambda\kappa}$ is defined as the rest, the latter being constructed by $\Delta^{\kappa}_{\lambda\mu}\left(\equiv F^{\kappa}_{\lambda\mu} - \left\{ {\kappa \atop \lambda\mu} \right\} \right)$. (This fact (3.3) is adapted to the definition $M_{\lambda\kappa} \equiv N^{i}_{\lambda} N^{j}_{\kappa} M_{ij}$). In (3.3), even if $R_{\lambda\kappa} = 0$, $M_{\lambda\kappa} \neq 0$ (i.e., $M_{ij} \neq 0$). This case corresponds to the compactification of the (y)-field considered by Gasperini[6].

Next, we shall take up the following field equation for the empty space :

$$S_{\nu\lambda} = 0 \qquad \text{...(3.4)}$$

where $S_{\nu\lambda}$ means the Ricci-tensor derived from the first curvature tensor $S^{\kappa}_{\nu\lambda\mu}$ defined by $C^{\kappa}_{\lambda\mu}$ (cf. Miron and Anastasiei[4] and Matsumoto[5]). Concerning this, it has been shown[7] that if $S_{\nu\lambda} = 0$, then $S^{\kappa}_{\nu\lambda\mu} = 0$ holds good, so that the field itself presents an almost Riemannian aspect, due to Brickell's theorem[8]. That is to say, the Finslerian field with $S_{\nu\lambda} = 0$ becomes almost Riemannian. Therefore, (3.4) seems somewhat unsuitable from our Finslerian viewpoint. Further, following the definition of $S^{\kappa}_{\nu\lambda\mu}$, the Ricci-tensor $S_{\nu\lambda}$ ($\equiv S^{\kappa}_{\nu\lambda\kappa}$) in our case based on (2.9) is calculated as

$$S_{\nu\lambda} = \tfrac{1}{2}\, \gamma^{\kappa\alpha} \left(\frac{\partial^2 h_{\nu\lambda}}{\partial y^\alpha\, \partial y^\kappa} - \frac{\partial^2 h_{\kappa\lambda}}{\partial y^\alpha\, \partial y^\nu} \right) \qquad \text{...(3.5)}$$

at least in the first order approximation with respect to $h_{\lambda\kappa}$. In (3.5), if $h_{\lambda\kappa} = \frac{1}{2} \frac{\partial^2 E(x, y)}{\partial y^\lambda\, \partial y^\kappa}$, then $S_{\nu\lambda} = 0$ (3.4) is identically satisfied. This is also unsuitable from a physical point of view[9].

The field equations (3.2) and (3.4) for the unified field may be reconsidered from the vector bundle-like standpoint as follows : In the total space, the adapted frame (2.6) has been set and the connection with the coefficients (2 4) has been introduced. This connection is made metrical for the metrical structure of the total space

$$G = G_{AB}\, d\zeta^A\, d\zeta^B = g_{\lambda\kappa}(x, y)\, dx^\kappa\, dx^\lambda + g_{ij}(x, y)\, \delta y^i\, \delta y^j. \qquad \text{...(3.6)}$$

In this case, the compactification process with respect to (2.8) is not taken into account and the frame (2.6) is assumed to be suitably adapted to the conditions $g_{\lambda i} = g_{i\lambda} = 0$. By straightforward calculations, the following six kinds of curvature tensors in the total space are obtained through the Ricci-identities with respect to the convariant derivatives (2.5) :

$$\mathscr{R}^{A}_{BCD} = (R^{\kappa}_{\lambda\mu\nu}, R^{l}_{i\lambda\mu}, P^{l}_{i\lambda k}, P^{\kappa}_{\lambda\mu k}, S^{\kappa}_{\lambda ij}, S^{l}_{jkl}). \quad ...(3.7)$$

As to the Ricci-tensors, they are given by, from (3.7),

$$\mathscr{R}_{AB} (\equiv \mathscr{R}^{C}_{ABC}) \equiv (R_{\lambda\mu} \equiv R^{\kappa}_{\lambda\mu\kappa}, \; P^{1}_{i\lambda} \equiv P^{\kappa}_{i\lambda k} - P^{2}_{\lambda i} \equiv P^{\kappa}_{\lambda k i},$$

$$S_{ij} \equiv S^{k}_{ijk}). \quad ...(3.8)$$

At this stage, following Mircon[4], we shall define the field equation for the total space in the form

$$\mathscr{R}_{AB} - \tfrac{1}{2}\mathscr{R}\, G_{AB} = \tau_{AB} \quad ...(3.9)$$

where the total scalar $\mathscr{R}$ $(\equiv \mathscr{R}_{AB}\, G^{AB})$ is given by $\mathscr{R} = R+S$ $(R = R_{\lambda\kappa}\, g^{\kappa\lambda}$, $S = S_{ij}\, g^{ij})$ and τ_{AB} means the eight-dimensional energy-momentum tensor depending on (x^{κ}, y^{i}). (3.9) is decomposed into the following four kinds of field equations[4] by (3.6) and (3 8) :

$$R_{\lambda\kappa} - \tfrac{1}{2}(R+S)\, g_{\lambda\kappa} = \tau_{\lambda\kappa}$$

$$P^{1}{}_{i\lambda} = \tau_{i\lambda},$$

$$P^{2}{}_{\lambda i} = -\tau_{\lambda i} \quad ...(3.10)$$

$$S_{ij} - \tfrac{1}{2}(S+R)\, g_{ij} = \tau_{ij}.$$

These equations can be obtained quite systematically without any special conditions on the spatial structure itself. Of course, they depend on the choice of adapted frame, so that different decompositions of (3.9) result in different equations. It turns out, therefore, that (3.2) and (3.4) are regarded as special cases of (3.10).

In (3.10), the most essential feature is the mixing of the $R_{\lambda\kappa}$-component representing the x-dependence and the S_{ij}-component representing the y-dependence. This means the interaction or coupling between the miro- and macro-degrees of freedom. The role of S in $(3.10)_1$ is compared to that of cosmological term, because S is considered an eight-dimensional effect (cf. Kerner[10]). It should be remarked that in the ordinary generalized Kaluza-Klein theory[6,10], such components as $P^{1}{}_{i\lambda}$ and $P^{2}{}_{\lambda i}$ (i.e., $\tau_{i\lambda}$ and $\tau_{\lambda i}$) are not taken into account from the beginning. It may be considered that (3.10) contains some more instructive information, so that its physical aspects should be investigated in future.

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