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PERIODIC SOLUTIONS OF A CERTAIN FOURTH ORDER DIFFERENTIAL EQUATION

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In this paper, we give sufficient conditions for the existence of periodic solutions of the nonautonomous equation (1.5).

1. INTRODUCTION

Consider the fourth-order constant coefficient differential equation :

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = 0. \qquad \text{..(1.1)}$$

As it was shown in Ezeilo[2] that if

$$a_4 > \tfrac{1}{4} a_2^2 \qquad \text{...(1.2)}$$

or

$$a_1 a_3 < 0, a_4 \neq 0 \qquad \text{...(1.3)}$$

then the auxiliary equation corresponding to (1.1) has no purely imaginary roots whatever. By the general theory; this, in turn, implies first of all that (1.1) has no periodic solution whatever other than $x = 0$, and secondly that the perturbed equation

$$x^{(4)} + a_1 \ddot{x} + a_2 \ddot{x} + a_3 \dot{x} + a_4 x = p(t) \qquad \text{...(1.4)}$$

in which p ($\neq 0$) is any continuous w-periodic function of t, has an w-periodic solution subject to (1.2) or (1.3).

In the literature, there are some extensions of (1.2) or (1.3) in one form or other of the existence result for eqns. (1.4) where some of a_1, ... a_4 are not constants[1,2,5]. The object of the present paper is to extend the result (1.2) for (1.4) to equation in which a_1, a_2, a_3 and a_4 are not all constants.

We shall be concerned with the equation

$$x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\dot{x})\ddot{x} + f_3(\dot{x}) + f_4(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \qquad \text{...(1.5)}$$

Where f_1, f_2, f_3, p are continuous functions depending only on the arguments shown and p is also assumed to be w-periodic in t. That is $p(t, x, y, z, u) = p(t + w, x,$

$y, z, u)$ for some $w > 0$ and for arbitrary t, x, y, z, u. We shall however require here that $f'_4(x)$ exists and is continuous for all x.

We shall establish here the following theorem :

Theorem—Suppose that

(i) there exists a constant $a_2 \geqslant 0$ such that

$$|f_2(y)| \leqslant a_2 \text{ for all } y \quad \ldots(1.6)$$

$$a_4 \equiv \inf_x f'_4(x) > \tfrac{1}{4} a_2^2 \quad \ldots(1.7)$$

(ii) there are constants $A_0 \geqslant 0$, $A_1 \geqslant 0$ such that

$$|p(t, x, y, z, u)| \leqslant A_0 + A_1(|y| + |z|) \quad \ldots(1.8)$$

for all t, x, y, z and u.

Then there exists a constant $\epsilon_0 > 0$ such that (1.5) has at least one w-periodic solution for all arbitrary f_1 and f_3 if $A_1 \leqslant \epsilon_0$.

It should be noticed that we get Theorem 3 given in Tejumola[4] under weaker conditions if we take $f_2(\dot{x}) = a_2$ and $f_4(x) = a_4 x$ in eqn. (1.5).

2. Preliminaries

As in Ezeilo[2], the proof will be by the Leray-Schauder technique, with the equation (1.5) embedded in the parameter-dependent equation :

$$x^{(4)} + \mu f_1(\ddot{x})\dddot{x} + \{(1-\mu)a_2 + \mu f_2(\dot{x})\}\ddot{x} + \mu f_3(\dot{x}) + (1-\mu)a_4 x$$
$$+ \mu f_4(x) = \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x}),\ 0 \leqslant \mu \leqslant 1. \quad \ldots(2.1)$$

Note that when $\mu = 1$ (2.1) reduces to the original eqn. (1.5). Also when $\mu = 0$ it reduces to the linear equation

$$x^{(4)} + a_2\ddot{x} + a_4 x = 0$$

which, in view of the condition (1.2), has no non-trivial w-periodic solution. Thus Theorem will follow from the usual fixed point considerations in Ressig *et al.*[3] if it can be shown that there is a constant D whose magnitude is independent of μ $(0 \leqslant \mu \leqslant 1)$ such that, if $x(t)$ is any w-perodic solution of (2.1), then

$$|x(t)| \leqslant D,\ |\dot{x}(t)| \leqslant D,\ |\ddot{x}(t)| \leqslant D,\ |\dddot{x}(t)| \leqslant D \quad \ldots(2.2)$$

for all $t \in [0, w]$. Note that the t-range here may be replaced by $[T, T+w]$ (arbitrary T) since we are dealing with a w-periodic $x(t)$.

Before proceeding to the actual verification of (2.2) we shall introduce some notations. Throughout what follows, D's with or without subscripts denote finite positive constants whose magnitudes depend on a_2 a_4, A_0, f_1 and f_3. The D's are all independent of μ. Finally a D without a subscript is not necessarily the same each time it occurs, but the numbered D's: D_0, D_1, ... retain a fiixed identity throughout.

3. Proof of Theorem

We shall take (2.1) in the more compact from

$$x^{(4)} + \mu\, f_1\,(\ddot{x})\,\dddot{x} + f_{2,\mu}\,(\dot{x})\,\ddot{x} + \mu f_3\,(\dot{x}) + f_{4,\mu}\,\mu\,(x) = \mu p\,(t,\, x,\, \dot{x},\, \ddot{x},\, \dddot{x})$$

$$(0 \leqslant \mu \leqslant 1) \qquad \ldots(3.1)$$

by setting

$$f_{2,\mu}\,(\dot{x}) = (1 - \mu)\, a_2 + \mu f_2\,(\dot{x})$$

$$f_{4,\mu}\,(x) = (1 - \mu)\, a_4\,(x + \mu\, f_4\,(x).$$

In what follows in the rest of this paper $x\,(t)$ is an arbitrary w-periodic solution of (3.1). It will now be shown that $x\,(t)$ satisfies (2.2) if A_1 is sufficiently small.

Our main tool in the verification of (2.2) for $x\,(t)$ is the function,

$$V\,(t) = \mu \int_0^{\ddot{x}} z\, f_1\,(z)\; dz + \ddot{x} \int_0^{\dot{x}} f_{2,\mu}\,(y)\, dy + \dddot{x}\,\ddot{x} + \dot{x}\, f_{4,\mu}\,(x) + \mu \int_0^{\dot{x}} f_3\,(y)\, dy.$$

An elementary differentiation gives

$$\dot{V}\,(t) = U_0 + \mu\, \ddot{x}\, p\,(t,\, x,\, \dot{x}\, \ddot{x},\, \dddot{x}) \qquad \ldots(3.2)$$

where

$$U_0 = \dddot{x}^2 + \dddot{x} \int_0^{\dot{x}} f_{2,\mu}\,(y)\, dy + \dot{x}^2 f'_{4,\mu}\,(x)$$

$$\geqslant \dddot{x}^2 - a_2\, |\,\dddot{x}\,|\;|\,\dot{x}\,| + a_4\, \dot{x}^2 \qquad \ldots(3.3)$$

by (1.6) and (1.7).

It is now not difficult to verify that

$$U_0 \geqslant D_0\,(\dddot{x}^2 + \dot{x}^2) \qquad \ldots(3.4)$$

for sufficiently small D_0. For, suppose for example that $D_0 < 1$, then, by (3.3),

$$U_0 - D_0\;(\dddot{x}^2 + \dot{x}^2) \geqslant (1 - D_0)\, \dddot{x}^2 - a_2\, |\,\dddot{x}\,|\;|\,\dot{x}\,| + (a_4 - D_0)\, \dot{x}^2$$

$$\geqslant \tfrac{1}{4}\,(1 - D_0)^{-1}\, U_1\, \dot{x}^2$$

where

$$U_1 = (4a_4 - a_2^2) - 4D_0(1 + a_4) + 4D_0^2.$$

However, by (1.7), $4a_4 - a_2^2 > 0$ and so U_0 is strictly positive if, say

$$0 < D_0 < \tfrac{1}{8}(4a_4 - a_2^2)(1 + a_4)^{-1}.$$

Thus, $U_0 - D_0(\dddot{x}^2 + \dot{x}^2) \geqslant 0$ if D_0 is sufficiently small, which gives (3.4) and hence by (3.2) and (1.8), leads to the estimate :

$$\begin{aligned} \dot{V}(t) &\geqslant D_0(\dddot{x}^2 + \dot{x}^2) - \{A_0 |\ddot{x}|\,|\dot{x}| + A_1 \ddot{x}^2\} \\ &\geqslant D_0 \dddot{x}^2 + (D_0 - A_1/2)\dot{x}^2 - A_0 |\ddot{x}| - \tfrac{3}{2} A_1 \ddot{x}^2 \qquad \ldots(3.5) \\ &\geqslant D_1(\dddot{x}^2 + \dot{x}^2) - \tfrac{8}{5} A_1 \ddot{x}^2 - D_2 \end{aligned}$$

for some D_1, D_2 if A_1 is taken sufficiently small.

Because of the (assumed) w-periodicity of $x(t)$, we have, on integrating (3.5), that

$$0 \geqslant D_1 \int_0^w (\dddot{x}^2 + \dot{x}^2)\,dt - \tfrac{8}{5} A_1 \int_0^w \ddot{x}^2\,dt - D_2 w. \qquad \ldots(3.6)$$

Combined with the inequality

$$\int_0^w \ddot{x}^2\,dt \leqslant \tfrac{1}{4} w^2 \pi^{-2} \int_0^w \dddot{x}^2\,dt \qquad \ldots(3.7)$$

which can be verified by substituting Fourier expansions of $\ddot{x}$ and $\dddot{x}$ in (3.6), (3.7) leads to the estimate

$$(D_1 - \tfrac{2}{5} w^2 \pi^{-2} A_1) \int_0^w \dddot{x}^2\,dt + D_1 \int_0^w \dot{x}^2\,dt \leqslant D_2 w.$$

Therefore, if A_1 is further fixed such that

$$A_1 w^2 \pi^{-2} \leqslant \tfrac{5}{4} D_1$$

as we assume henceforth, then

$$D_1 \int_0^w (\dddot{x}^2 + \dot{x}^2)\,dt \leqslant 2 D_2 w. \qquad \ldots(3.8)$$

In particular

$$\int_0^w \dddot{x}^2\,dt \leqslant D_3.$$

Considering now identity

$$\ddot{x}(t) = \ddot{x}(T_1) + \int_{T_1}^{t} \dddot{x}^2(s)\, ds$$

with T_1 fixed (as is possible in view of the periodicity condition $\dot{x}(0) = \dot{x}(w)$) such that $\ddot{x}(T_1) = 0$, we have that

$$\max_{0<t\leqslant w} |\ddot{x}(t)| \leqslant \int_0^w |\ddot{x}(s)|\, ds \leqslant w^{1/2} \left(\int_0^w \dddot{x}^2(s)\, ds\right)^{1/2}$$

by Schwarz's inequality. Thus (3.8) implies that

$$\max_{0<t<w} |\ddot{x}(t)| \leqslant D_3^{1/2} w. \qquad \ldots(3.9)$$

From this, on referring to the identity

$$\dot{x}(t) = \dot{x}(T_2) + \int_{T_2}^{t} \ddot{x}(s)\, ds.$$

With T_2 chosen such that $\dot{x}(T_2) = 0$ (the choice being possible in view of the periodicity condition $x(0) = x(w)$), we have that

$$\max_{0<t\leqslant w} |\dot{x}(t)| \leqslant D_3^{1/2} w^2. \qquad \ldots(3.10)$$

To obtain an estimate for $|x(t)|$ first note that, because of the w-periodicity of $x(t)$, integration of both sides of (3.1) yields the result

$$\int_0^w -\{f_{4,\mu}(x) - \mu p(t, x, \dot{x}, \ddot{x}\, \dddot{x})\}\, dt = \mu \int_0^w f_3(\dot{x})\, dt$$

or indeed, in view of (3.10), that

$$\left| \int_0^w \{f_{4,\mu}(x) - \mu p(t, x\, \dot{x}, \ddot{x}, \dddot{x})\}\, dt \right| \leqslant D_4 \qquad \ldots(3.11)$$

by (1.8), (3.9) and (3.10)

$$|\mu p(t, x, \dot{x}, \ddot{x}, \dddot{x})| \leqslant D_3^{1/2} w$$

for some D_3. Also, since $a_4 > 0$ and $f_4(x)\operatorname{sgn} x \to \infty$ as $|x| \to \infty$ (by (1.7)), there clearly exists D_6 independent of μ such that

$$|f_{4,\mu}(x) - \mu p(t, x, \dot{x}, \ddot{x}, \dddot{x})| \geqslant 2D_4 w^{-1} \qquad \ldots(3.12)$$

if $|x(t)| \geqslant D_5$ for all $t \in [0, w]$. It is thus clear that $|x(T_3)| \leqslant D_5$ for some T_3, as otherwise, by (3.12), the left handside of (3.11) would be not less in magnitude than $2D_4$. The result that $|x(T_3)| \leqslant D_5$ combined with (3.10) to yield the required boundedness estimate for x:

$$\max_{0<t\leqslant w} |x(t) \leqslant D_4 + D_3^{1/2} w^3. \qquad \ldots(3.13)$$

It remains now to obtain estimate for $|\dddot{x}(t)|$ in order to complete our verification of (2.2). Multiplying (3.1) by $x^{(4)}$ and integrating from $t = 0$ to $t = w$ we obtain

$$\int_0^w (x^{(4)})^2\, dt = -\mu \int_0^w f_1(\ddot{x})\, \dddot{x}\, x^{(4)}\, dt + \int_0^w x^{(4)}\, Q dt \qquad \ldots(3.14)$$

where

$$Q = \mu p(t, x\ \dot{x}, \ddot{x}, \dddot{x}) - \{f_{2,\mu}(\dot{x})\,\ddot{x} - \mu f_3(\dot{x}) - f_{4,\mu}(x)\}$$

satisfies

$$|Q| \leqslant D_6 \qquad \ldots(3.15)$$

by (1.6), (1.8), (3.9), (3.10) and (3.13). But by (3.9), $|f_1(\ddot{x})| \leqslant D_7$, so that

$$\left| -\mu \int_0^w f_1(\ddot{x})\, \dddot{x}\, x^{(4)}\, dt \right| \leqslant D_7 \left(\int_0^w (x^{(4)})^2\, dt\right)^{1/2}$$

by (3.8). Thus from (3.14) and (3.15), we have that

$$\int_0^w (x^{(4)})^2\, dt \leqslant D_8 \left(\int_0^w (x^{(4)})^2\, dt\right)^{1/2}$$

and hence that

$$\int_0^w (x^{(4)})^2\, dt \leqslant D_9$$

for some D_9. The result

$$\max_{0<t\leqslant w} |\dddot{x}(t)| \leqslant w^{1/2} D_9^{1/2} \qquad \ldots(3.16)$$

now follows readily.

The result (3.9), (3.10), (3.13) and (3.16) fully verify (2.2) for the arbitrary chosen w-periodic solution $x(t)$ of (3.1) if the A_1 in (1.8) is sufficiently small. This now completes the proof of the Theorem.

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