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CUBIC TRANSFORMATIONS OF FINSLER SPACES AND n FUNDAMENTAL FORMS OF THEIR HYPERSURFACES

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If $L(x, y)$ is the metric function of a Finsler space $F^n = (M^n, L)$ and β is a one-form $b_i(x) y^i$ in F^n, then the trannsformation $L \to L^* = f(L, \beta)$ known as β-change has been introduced by Shibata[9]. In the present paper we consider a particular case of β changes known as cubic transformation given by $L \to L^* = (L^3 + \beta^3)^{1/3}$. The relation between n fundamental forms of tangent Riemannian hypersurface of (M^n, L) and (M^n, L^*) has been obtained.

1. INTRODUCTION

Let $F^n = (M^n, L)$ be an n dimensional Finsler space with fundamental metric function $L(x, y)$. In general $L(x, y)$ is a function of point $x (= x^i)$ and element of support $y (= y^i)$ and positively homogeneous of degree one in y. Let (M^n, L') and (M^n, L'') be Finsler spaces whose metric functions $L'(x, y)$ and $L''(x, y)$ are obtained from L by the relations

$$L' = L + \beta \qquad ...(1.1)$$

$$L''^2 = L^2 + \beta^2 \qquad ...(1.2)$$

where $\beta = b_i y^i$, $b_i(x)$ is a component of a covariant vector which is a function of position alone. These two transformations have been introduced by Matsumoto[1] which have the geometrical properties stated therein. Generalizing these transformations, Chibata[9] has studied the properties of Finsler space (M^n, L^*) whose metric function $L^*(x, y)$ is obtained from L by the relation $L^*(x, y) = f(L, \beta)$, where f is positively homogeneous of degree one in L and β. This change of metric function is called a β-change. A particular case of β-changes is a cubic transformation of metric function given by

$$L^{*3} = L^3 + \beta^3. \qquad ...(1.3)$$

The n fundamental forms of a Riemannian hypersurface of a Riemannian space have been defined and their properties have been studied by Rund[7,8]. Prasad[5] has obtained the relation in n fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L') whose metric functions are related by (1.1). He[6] also obtained the relation in n fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L'') whose metric functions are related by (1.2). In this paper we shall obtain the relation in n fundamental forms of tangent Riemannian hypersurfaces of $F^n = (M^n, L)$ and $F^{*n} = (M^n, L^*)$ whose metric functions are related by (1.3).

2. The Relation Between v-Curvature Tensors of F^n and F^{*n}

Throughout this paper the quantities corresponding to the Finsler space F^{*n} will be written by putting '*'. For the cubic transformation (1.3) the relation between the angular metric tensors h_{ij} and h^*_{ij} of F^n and F^{*n} will be given by

$$h^*_{ij} = p\, h_{ij} + 2pq\, m_i\, m_j \qquad \ldots(2.1)$$

where $p = L\, L^{*-1}$, $q = \beta\, L^{*-1}$, $m_i = q\, l_i - pb_i$ and $l_i = \partial_i L$. Thus the metric tensor g^*_{ij} of F^{*n} and its recipocal g^{*ij} are given by

$$g^*_{ij} = pg_{ij} + pq^3\, l_i\, l_j - p^2\, q^2\, (l_i\, b_j + l_j\, b_i)$$
$$+ q\, (l + p^3)\, b_i\, b_j \qquad \ldots(2.2)$$
$$g^{*ij} = p^{-1}\, g^{ij} - pq^3\, \lambda\, (p + qb^2)\, l^i\, l^j + q^2\, \lambda\, (l^i\, b^j + l^j\, b^i)$$
$$- 2pq\, \lambda\, b^i\, b^j \qquad \ldots(2.3)$$

where we put

$$b^i = g^{ij}\, b_j,\ l^i = g^{ij}\, l_j,\ b^2 = g^{ij}\, b_i\, b_j,$$
$$\lambda^{-1} = p^3 - q^3 + 2p^2\, qb^2.$$

The differentiation of (2.2) with respect to y^k gives the (h) hv-torsion tensor $C^*_{ijk} = (\partial\, g^*_{ij}\, /\partial y^k)/2$ of F^{*n}:

$$C^*_{ijk} = pC_{ijk} + pq^2\, (h_{ij}\, m_k + h_{ik}\, m_j + h_{jk}\, m_i)/2L$$
$$- (2p^3 - 1)\, (p/L)\, m_i\, m_j\, m_k. \qquad \ldots(2.4)$$

Paying attention to $m_i\, y^i = 0$, $h_{ij}\, y^i = 0$, $C_{ijk}\, y^i = 0$, we obtain

$$C^{*h}_{ij} = C^*_{ijk}\, g^{*hk} = C^h_{ij} + (q^2/2L)\, (h^h_i\, m_j + h^h_j\, m_i)$$

(equation continued on p. 244)

$$+ pq\lambda(ql^h - 2pb^h) C_{ij} + p^2 q^3 (p + qb^2)(\lambda/2L) h_{ij} l^h$$
$$- (pq^2 \lambda/2L) h_{ij} b^h - (q\lambda/L)[p^2 (2p^3 - l)(p + qb^2)$$
$$+ q^3] m_i m_j l^h + (p\lambda/L) m_i m_j b^h \quad ...(2.5)$$

where

$$C_{ij} = C_{ijk} b^k.$$

We shall now find the v-curvature tensor S^*_{hijk} which is given by $S^*_{hijk} = C^{*m}_{ij} C^*_{hkm} - C^{*m}_{ik} C^*_{hjm}$. By the use of (2.4) and (2.5) we get

$$S^*_{hijk} = p S_{hijk} + C_{ij} d_{hk} + C_{hk} d_{ij} - C_{ik} d_{hj} - C_{hj} d_{ik}$$
$$+ h_{ij} E_{hk} + h_{hk} E_{ij} - h_{ik} E_{hj} - h_{hj} E_{ik} \quad ...(2.6)$$

where

$$d_{hk} = (p^2 \lambda/L) m_h m_k - pq^3 C_{hk} - (p^2 q^2 \lambda/4L) h_{hk} \quad ...(2.7)$$
$$E_{hk} = (pq^2\lambda/4L^2)[q^2 (2p^3 + 1) - 2p^5 b^2] m_h m_k$$
$$- (pq^4 \lambda/8L^2)(q^2 - p^2 b^2) h_{hk} - (p^2 q^2 \lambda/4L) C_{hk}. \quad ...(2.8)$$

Remark : It is to be noted that if $L(x, y)$ is the metric function of a Riemannian space i.e. $L(x, y) = ((a_{ij}(x) y^i y^j)^{1/2} = \alpha$ then $L^*(x, y) = (\alpha^3 + \beta^3)^{1/3}$. Thus F^{*n} is a Finsler space with an (α, β) metric. It is known that a Finsler space with an α, β-metric is semi-C-reducible[4]. In order to clearify this fact in this case, calculating from (2.4) we have

$$C^*_{ijk} = U(h^*_{ij} C^*_k + h^*_{jk} C^*_i + h^*_{ik} C^*_j)/(n + 1)$$
$$+ VC^*_i C^*_j C^*_k / C^{*2} \quad ...(2.9)$$

where

$$U = (n + 1) q^2 A/2L, V = -(1 + q^3) C^{*2} A^3/L^*, C^{*2} = g^{*ij} C^*_i C^*_j,$$

and

$$A = 2L/\{(n + 3\lambda) q^2 - 2p^2 b^2 \lambda\}.$$

3. Hypersurface of (M^n, L)

Let (M^{n-1}, L) be a hypersurface of (M^n, L) given by the equation

$$x^i = x^i(u^\alpha). \quad ...(3.1)$$

Let us suppose that the functions (3.1) are at least of class C^3 in u^α and the projection factors $B^j_\alpha = \partial x^j/\partial u^\alpha$ are such that their matrix has maximal rank $n-1$. The fundamental metric function $\mathbf{L}(u, v)$ of the hypersurface is given by

$$L(u^\alpha, v^\alpha) = L(x^i(u^\alpha),\ B^i_\alpha\ v^\alpha)$$

where v^α is the element of support for the hypersurface for which

$$y^i = = B^j_\alpha\ v^\alpha. \qquad \ldots(3.2)$$

Thus if l^α denote the normalized vector along the element of support then $l^i = B^i_\alpha\ l^\alpha$. If $g_{hj}(x, y)$ denote the metric tensor of (M^n, L), the induced metric tensor of $(M^{n-1}, \mathbf{L})$ is given by

$$g_{\alpha\beta}(u, v) = g_{hj}(x, y)\ B^h_\alpha\ B^j_\beta\,. \qquad \ldots(3.3)$$

The inverse of (3.3) is denoted by $g^{\alpha\beta}(u, v)$ by means of which we define the quantities

$$B^\alpha_i\ (u, v) = g^{\alpha\beta}(u, v)\, g_{ij}(x, y)\ B^j_\beta\,. \qquad \ldots(3.4)$$

The unit normal vector $N^j(u, v)$ of $(M^{n-1}, \mathbf{L})$ is determined by the relations

$$g_{hj}(x, y)\, B^h_\beta\ N^j(u, v) = 0,\ g_{hj}(x, y)\, N^h(u, v)\, N^j(u, v) = 1. \qquad \ldots(3.5)$$

We have the following identities from (3.3), (3.4) and (3.5) :

$$B^\alpha_j\ B^j_\beta = \delta^\alpha_\beta\,,\ B^j_\alpha\ B^\alpha_h + N^j N_h = \delta^j_h \qquad \ldots(3.6)$$

where

$$N_i = g_{ij}(x, y)\, N^j.$$

If $C_{hjk}(x, y)$ denotes the (h) hv-torsion tensor of $(M^n\ L)$, the induced (h) hv-torsion tensor $C_{\alpha\beta\gamma}(u, v)$ of $(M^{n-1}, \mathbf{L})$ is given by

$$C_{\alpha\beta\gamma}(u, v) = C_{hjk}(x, y)\ B^h_\alpha\ B^j_\beta\ B^k_\gamma \qquad \ldots(3.7)$$

from which we obtain

$$C^\alpha_{\beta\gamma} = B^\alpha_i\ C^i_{jk}\ B^j_\beta\ B^k_\gamma\,. \qquad \ldots(3.8)$$

The relative v-covariant derivative of the projection factor B^i_α with respect to the induced Cartan connection $IC\,\Gamma$ is given by[3]

$$B^i_{\beta|\gamma} = - B^i_\alpha \; C^\alpha_{\beta\gamma} + C^i_{hk} \; B^h_\beta \; B^k_\gamma . \qquad ...(3.9)$$

This tensor is normal to (M^{n-1}, L). Therefore we may write

$$B^i_{\beta|\gamma} = M_{\beta\gamma} \; N^i . \qquad ...(3.10)$$

From (3.9) it is clear that $M_{\beta\gamma}$ is symmetric in β and γ and it may be written as

$$M_{\beta\gamma} = C_{ijk} \; N^i \; B^j_\beta \; B^k_\gamma . \qquad ...(3.11)$$

The tangent vector space M^{n-1}_x to M^{n-1} at every point x^i $(= u^\alpha)$ of the hypersurface is considered as the Riemannian space (M^{n-1}_x, g_x) with the Riemannian metric $g_x = g_{\alpha\beta}(u, v) \, dv^\alpha \, dv^\beta$. The components of the (h) hv-torsion tensor $C^\alpha_{\beta\gamma}$ will be the Christoffel symbols associated with g_x. If M^n_x is the tangent vector space to M^n at x^i $(= u^\gamma)$, then (M^{n-1}_x, g_x) will be the hypersurface of (M^n_x, g_x) given by (3.2), where $g_x = g_{ij}(x, y) \, dy^i \, dy^j$ is the Riemannian metric on M^n_x. The quantities $M_{\beta\gamma}$ given by (3.11) will be considered as the coefficients of the second fundamental form of the tangent Riemannian space (M^{n-1}_x, g_x).

In general the coefficients of the rth fundamental form of (M^{n-1}, g_x) are defined as[7]

$$C_{(1)\alpha\beta} = g_{\alpha\beta}, \; C_{(2)\alpha\beta} = M_{\alpha\beta}, \; C_{(r)\alpha\beta} = C_{(r-1)\alpha\delta} \; M^\delta_\beta \; (2 \leqslant r \leqslant n) \qquad ...(3.12)$$

where

$$M^\delta_\beta = g^{\alpha\delta} \, M_{\alpha\beta} .$$

4. The n Fundamental Forms of a Hypersurface of (M^n, L^*)

Let (M^{n-1}, L^*) be a hypersurface of $(M^n \; L^*)$ given by the same equation (3.1). It is to be noted that a unit normal vector N^i to (M^{n-1}, L^*) is not necessarily normal to (M^{n-1}, L^*). Paying attention to $l_i \, N^i = 0$, from (2.2) we have

$$g^*_{ij} \; B^i_\alpha \; N^j = (b_i N^i) \, \{q \, (1 + p^3) \, b_\alpha - p^2 \, q^2 \, l_\alpha\} \qquad ...(4.1)$$

$$g^*_{ij} \; N^i \, N^j = p + q \, (1 + p^3) \, (b_i \, N^i)^2 \qquad ...(4.2)$$

where $b_\alpha = b_l B^l_\alpha$. If b_i is tangential to the hypersurface (M^{n-1}, L), that is, $b_i N^i = 0$, then we have

$$N^{*i} = p^{-1/2} N^i \qquad \text{...(4.2)}$$

where we have chosen a normal vector N^{*i} to (M^{n-1}, L^*) in the same direction as N^i. Hence we have the following :

Theorem 4.1—Let (M^n, L^*) be a Finsler space obtained from a Finsler space (M^n, L) by the transformation (1.3). If (M^{n-1}, L^*) and (M^{n-1}, L) are the hypersurface of these spaces given by the same equation (3.1) and b_i is tangential to the hypersurface (M^{n-1}, L), then the vector normal to (M^{n-1}, L) is also normal to (M^{n-1}, L)

Now we establish the following :

Theorem 4.2—Let (M^n, L^*) be a Finsiler space obtained from a Finsler space (M^n, L^*) by the transformation (1.3). Let (M^{n-1}, L^*) and (M^{n-1}, L) be the hypersurfaces of (M^n, L^*) and (M^n, L) given by the same equation (3.1). If b_i is tangential to the hypersurface(M^{n-1}, L), and (M^n_x, g_x), $(M^n_x\, g^*_x)$, (M^{n-1}, g^*_x), (M^{n-1}, g_x), are the tangent Riemannian space to (M^n, L), (M^n, L^*) (M^{n-1}, L) (M^{n-1}, L^*) respectively, then we have the following :

(i) The second fundamental forms of $(M^{n-1}_x\, g_x)$ and $(M^{n-1}_x, g^*_x)$are proportional.

(ii) Every asymptotic direction of (M^{n-1}_x, g_x) is also an asymptotic direction of (M^{n-1}_x, g^*_x).

(iii) The rth fundamental tensors of (M^{n-1}_x, g_x) and (M^{n-1}_x, g^*_x) are related by

$$C^*_{(r)\alpha\beta} = p^{(2-r)/2} \left[C_{(r)\alpha\beta} - \sum_{m=2}^{r-1} X_{(m)\beta}\, Q_{(r+1-m)\alpha}\right] (3 \leqslant r \leqslant n) \qquad \text{...(4.4)}$$

where

$$X_{(m)\alpha} = P \sqrt{2q\lambda}\; C_{(m)\alpha\delta}\, b^\delta \; (2 \leqslant m \leqslant n-1) \qquad \text{...(4.5)}$$

$$Y_{(m)} = P \sqrt{2q\lambda}\; X_{(m)\delta}\, b^\delta \; (2 \leqslant m \leqslant n-1) \qquad \text{...(4.5b)}$$

$$Q_{(2)\alpha} = X_{(2)\alpha} \qquad \text{.. (4.5c)}$$

$$Q_{(r)\alpha} = X_{(r)\alpha} - \sum_{m=2}^{r-1} Y_{(m)}\, Q_{(r+1-m)\alpha} \; (3 \leqslant r \leqslant n-1). \qquad \text{...(4.5d)}$$

PROOF : (i) Since $b_i N^i = 0$ implies $m_i N^i = 0$, from (2.4), (3.11) and (4.3) it follows that

$$M^*_{\beta\gamma} = p^{1/2} M_{\beta\gamma}. \qquad \ldots(4.6)$$

This proves (i).

(ii) A direction t^α for which $M_{\alpha\beta} t^\alpha t^\beta = 0$ is said to be an asymptotic direction. In view of this definition and (4.6) we get (ii).

(iii) The validity of relation (4.4) is established by induction. Since C_{ijk} is an indicatory tensor, from (3.11) it follows that $M_{\beta\gamma} l^\gamma = 0$. Hence from (3.12) we have

$$C_{(r)\beta\gamma} l^\gamma = 0 = C_{(r)\beta\gamma} l^\beta \; (2 \leqslant r \leqslant n). \qquad \ldots(4.7)$$

Hence from (4.5) (a), (c) and (d) we get

$$X_{(r)\alpha} l^\alpha = 0, \; Q_{(r)\alpha} l^\alpha = 0 \; (2 \leqslant r \leqslant n). \qquad \ldots(4.8)$$

Since $g^{*\alpha\beta} = g^{*ij} B^\alpha_i B^\beta_j$, from (2.2) and (4.6) we get

$$M^{*\delta}_{\beta} = p^{-1/2} [M^\delta_\beta + pq^2 \lambda M_{\alpha\beta} b^\alpha l^\delta - p\sqrt{2q\lambda} X_{(2)\beta} b^\delta] \qquad \ldots(4.9)$$

where $b^\alpha = B^\alpha_i b^i$. The relations (3.12), (4.5a) (4.6), (4.8) and (4.9) yield

$$C^*_{(3)\alpha\beta} = C_{(3)\alpha\beta} - X_{(2)\alpha} X_{(2)\beta}. \qquad \ldots(4.10)$$

From (4.5c) and (4.10) it is evident that (4.4) holds for $r = 3$. For a given fixed value of the integer s with $3 \leqslant s \leqslant n - 1$ we have

$$C^*_{(s+1)\alpha\beta} = C^*_{(s)\alpha\delta} M^{*\delta}_{\beta}. \qquad \ldots(4.11)$$

Now let us suppose that (4.4) is valid for $s = 3, 4, 5, \ldots r$. so that we can write (4.11) in the form

$$C^*_{(s+1)\alpha\beta} = p^{(3-s)/2} [C_{(s)\alpha\delta} - \sum_{m=2}^{s-1} X_{(m)\delta} Q_{(s+1-m)\alpha}] M^{*\delta}_{\beta}$$

which in view of (4.5), (4.7), (4.8) and (4.9) gives

$$C^*_{(s+1)\alpha\beta} = p^{(2-s)/2} [C_{(s+1)\alpha\beta} - \sum_{m=2}^{s-1} X_{(m+1)} Q_{(s+1-m)\alpha}$$

(equation continued on p. 249)

$$- X_{(2)\beta} \{X_{(s)\alpha} - \sum_{m=2}^{s-1} Y_{(m)} Q_{(s+1-m)\alpha}\}]$$

$$= p^{(2-s)/2} [C_{(s+1)\alpha\beta} - \sum_{m=2}^{s} X_{(m)\beta} Q_{(s+2-m)\alpha}].$$

This shows that (4.4) is valid for $r = (s + 1)$, which completes the proof of (iii).

Remark 1 : If $q\lambda < 0$, the quantities $X_{(m)\alpha}$, $Y_{(m)}$ and $Q_{(m)\alpha}$ defined in (4.5) are complex quantities. However the quantities $C^{*}_{(r)\alpha\beta}$ are always real.

Remark 2 : Theorem (4.1) and Theorem 4.2 are also valid for general β-changes.

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