Indian J. pure appl. Math., 20 (3) : 250-259, March 1989

COPURE-INJECTIVE MODULES*

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(*Received* 10 *November* 1987; *after revision* 22 *March* 1988)

In this paper we introduce the notion of a copure-injective module and study its properties over special types of rings. Our main structure theorem asserts that over an arbitrary ring R, the copure-injective right R-modules are precisely the direct summands of direct products of cofinitely related right R-modules. Over a commutative classical ring R the copure-injective R-modules are precisely the pure-injective R-modules. If R is a commutative ring then every pure-injective R-module is copure-injective. Over a Dedekind domain R, every copure homomorphic image of a copure-injective R-module is copure-injective. Finally we derive the analogue of Schanuel's lemma for copure short exact sequences and copure-injective modules.

Throughout this paper by a ring R we mean an associative ring with identity and by an R-module we mean an unitary right R-module while mod-R stands for the category of all right R-modules and R-homomorphisms.

Definitions 1—(i) An R-module M is said to be 'finitely embedded' (Vamos[19], p. 643) (or 'cofinitely generated' (Jans[10], p. 588)) if $E(M) = E(S_1) \oplus E(S_2) \oplus \ldots \otimes E(S_n)$, where $S_1, S_2, \ldots, S_n$ are simple R-modules (here $E(X)$ denotes the injective hull of an R-module X).

(ii) An R-module M is said to be 'cofree' (Hiremath[7], Definition 6) if M is isomorphic to $\Pi\, \{E(S_\alpha) : S_\alpha$ is a simple R-module, $\alpha \in \Lambda\}$ for some index set Λ.

(iii) An R-module A is said to be 'cofinitely related' (Hiremath[7], Definition 14) if there is an exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B cofinitely generated, cofree and C cofinitely generated.

(iv) A short exact sequence of R-modules is said to be 'copure' (Hiremath[8], Definition 3) if every cofinitely related R-module is injective relative to this sequence.

(v) A ring R is said to be 'right co-noetherian' (Jans[10], p. 588) if every homomorphic image of a cofinitely generated R-module is cofinitely generated.

(vi) A short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is said to be 'pure' (Warfield[21], p. 703) if for every left R-module M, the induced map $f \otimes I_M : A \otimes_R M \to B \otimes_R M$ of abelian groups is a monomorphism.

*This work is based on the author's doctoral dissertation and was supported by the University Grants Commission and Madurai Kamaraj University.

(vii) An R-module is said to be 'pure-injective' (Warfield[21], p. 703) if it is injective relative to each pure short exact sequence of R-modules.

Definition 2—An R-module is said to be copure-injective if it is injective relative to every copure short exact sequence of R-modules.

From the definition we have following easy consequences.

Proposition 3—(i) Every injective R-module is copure-injective.

(ii) Every cofinitely related R-module is copure-injective.

(iii) A copure short exact sequence $0 \to A \to B \to C \to 0$ of R-modules with A copure-injective, splits.

(iv) Every direct summand of a copure-injective R-module is copure-injective.

(v) Every direct product of copure-injective R-modules is copure-injective.

Remark 4 : A copure-injective R-module need not be injective.

Example—Let $G = Z(n)$ be a cyclic group of order n. Since G is finite it is cofinitely generated and hence, cofinitely related as a Z-module (Hiremath[7], p, 5 and Proposition 17). So, by Proposition 3 (ii), G is copure-injective as a Z-module. But we know that G is not injective as a Z-module.

Proposition 5—For a ring R the following conditions are equivalent:

(i) R is a right V-ring;

(ii) every cofinitely related R-module is injective;

(iii) every short exact sequence of R-modules is copure;

(iv) every copure-injective R-modules is injective.

POOOF : (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow by Proposition 5 of Hiremath[8]. (iii) $\Rightarrow$ (iv) is obvious and (iv) $\Rightarrow$ (ii) follows by Proposition 3 (ii).

Remark : A direct sum of (copure—) injective R-modules need not be copure-injective (see example (i) below). We know that over a right noetherian ring R every direct sum of injective R-modules is injective (Faith[3], Proposition 6.5). But over a right noetherian ring R (even when R is a Dedekind domain) a direct sum of copure-injective R-modules need not be copure-injective (see example (ii)).

Examples—(i) Let $R = \prod_{n=1}^{\infty} Z/p_n Z$, where $\{p_1, p_2, \ldots, p_n, \ldots\}$ is the set of all positive primes. Since R is a Von Neumann regular ring, R is a V-ring by Theorem 6 of Rosenberg and Zelinsky[16]. Let, for $n = 1,2, \ldots$, $\mathcal{P}_n = \prod_{i=1}^{n} Z/p_i Z$. Then $\mathcal{P}_1 \subsetneq$

$\mathcal{P}_2 \underset{+}{\subset} \cdots \underset{+}{\subset} \mathcal{P}_n \underset{+}{\subset} \cdots$ is a strictly ascending chain of ideals of R. Then, by Proposition 6.5 of Faith[3], $E = \overset{\infty}{\underset{n=1}{\oplus}} E(R/\mathcal{P}_n)$ is not injective and so, not copure-injective by Proposition 5, as R is a V-ring.

(ii) Let $G = \overset{\infty}{\underset{n=1}{\oplus}} Z/p_n Z$ where $\{p_1, p_2, \ldots, p_n, \ldots\}$ is the set of all positive primes. Since each $Z/p_n Z$ is finite, it is cofinitely related as a Z-module by the remark on p. 5 and Proposition 17 of Hiremath[7] and so, by Proposition 3 (ii), each $Z/p_n Z$ is copure-injective as a Z-module. Clearly G is a pure submodule of the Z-module $H = \overset{\infty}{\underset{n=1}{\Pi}} Z/p_n Z$. Hence, by Proposition 12 of Hiremath[8], G is a copure submodule of the Z-module H as Z is a neotherian ring. If G were copure-injective, then by Proposition 3 (iii), G would be a direct summand of H which is absurd. So G is not copure-injective as a Z-module.

We recall (Faith[4], p. 254) that a category $\mathscr{C}$ is 'locally small' if the equivalence class of subobjects of any object of $\mathscr{C}$ is a set and $\mathscr{C}$ is 'colocally small' if the dual category $\mathscr{C}^*$ of $\mathscr{C}$ is locally small. Since the category mod-R is colocally small (Faith[4], Exercise 5.27.4), the isomorphisms closed class of cyclic R-modules, distinct up to isomorphisms, is a set. Hence the isomorphism closed class $\mathscr{S}$ of simple R-modules, distinct up to isomorphisms, is a set. Since mod-R is a locally small category (Faith[4], Exercise 5.27.4), the isomorphism closed class of subobjects of the R-module $A = \Pi\ \{E(S) : S \in \mathscr{S}\}$ (distinct upto isomorphisms) is a set. Since any cofinitely generated (and hence, cofinitely related) R-module is isomorphic to submodule of A, the isomorphim closed class C_r^* of cofinitely related R-modules, distinct up to isomorphisms, is a set.

We now prove that for any ring R, there are enough copure-injective R-modules in the sense that every R-module can be embedded in a copure-injective R-module as a copure submodule by applying Proposition 3 to :

Theorem 7—Every R-module can be embedded ais a copure submodule in a direct product of cofinitely related R-modules.

PROOF : Let C_r^* be the isomorphism closed class of all cofinitely related R-modules distinct up to isomorphisms. We have seen above that C_r^* is a set.

Let A be any R-module and let

$$Q = \Pi\ \{M^{\mathrm{Hom}_R(A, M)} : M \in C_r^*\}.$$

Define a map

$$\phi : A \to Q \text{ by}$$

$$\phi(a) = ((f(a))_{f \in \mathrm{Hom}_R(A,M)})_{M \in C_r^*}, a \in A.$$

Clearly ϕ is an R-homomorphism. We claim that ϕ is a copure monomorphism. Let $0 \neq a \in A$. Let $\mathfrak{M}$ be a maximal right ideal of R containing $\mathrm{ann}_R(a)$. Then $S = R/\mathfrak{M}$ is cofinitely generated and hence $E(S) \in C_r^*$. Define a map $\alpha : aR \to S$ by $\alpha(ar) = r + \mathfrak{M}$, $r \in R$. Then α extends to a homomorphism $f: A \to E(S)$. Since $f(a) = 1 + \mathfrak{M} \neq 0$, $\phi(a) \neq 0$, proving that ϕ is a monomorphism.

To prove that ϕ is copure, let $M \in C_r^*$ and $f \in \mathrm{Hom}_R(A, M)$. Then clearly $f = p\phi$ where $p : Q \to M$ is the composition $\pi_f \circ \pi_M$ of the usual projections π_M and π_f.

We now have the following structure theorem for copure-injective modules.

Theorem 8—An R-module M is copure-injective if and only if it is a direct summand of a direct product of cofinitely related R-modules.

PROOF : Necessity follows from Theorem 7 and Proposition 3 (iii) and sufficiency from (ii), (iv) and (v) of Proposition 3.

We now compare copure-injectivity with pure-injectivity. For this we first observe that from the adjoint isomorphism of Hom and $\otimes$ one can easily deduce that for a left R-module M, the canonical R-module $M^* = \mathrm{Hom}_Z(M, Q/Z)$ is pure-injective.

Proposition 9—Over a commutative ring R every pure-injective R-module is copure-injective.

PROOF : This follows from the fact that for a commutative ring R, copurity implies purity (Hiremath[9], Proposition 13).

Remark 10 : A pure--injective R-module need not be copure-injective.

Example—Cozzens[2] has constructed a ring $R = k[x, D]$ of differentiable polynomials in a single indeterminate x over a universal field k with a derivation D (where multiplication is given by $ax = xa + D(a)$, $a \in k$). Cozzens has proved that R is a right V-ring but not a field. Since x is not invertible in R, Rx is not a pure left ideal of R and hence R/Rx is not flat as a left R-module. So by the Corollary on p. 131 of Lambek[12], $(R/Rx)^* = \mathrm{Hom}_Z(R/Rx, Q/Z)$ is not injective as an R-module whence, by Proposition 5, $(R/Rx)^*$ is not copure-injective. But $(R/Rx)^*$ is pure-injective.

Proposition 11—If R is a commutative (co-) noetherian ring then every copure-injective R-module is pure-injective.

PROOF : Since over a commutative (co-) noetherian ring R, every pure submodule of an R-module is copure (Hiremath[8], Proposition 12), the proposition follows.

We now prove a proposition which dualizes (i) $\Leftrightarrow$ (iv) of Proposition 5 :

Proposition 12—A ring R is a Von Neumann regular if and only if every pure-injective R-module is injective.

PROOF : Since for a Von Neumann regular ring R, every short exact sequence of R-modules is pure, by Theorem 11.24 of Faith[4] the 'only if part follows.

For the 'if' part, suppose that every pure-injective R-module is injective. To prove that R is Von Neumann regular, we need only prove, by Theorem 11.24 of Faith[4], that every left R-module M is flat. Indeed, since the R-module $M^* = \text{Hom}_Z(M, Q/Z)$ is pure-injective, it is injective by hypothesis. It then follows. from the Corollory on on p. 131 of Lambek[12] that M is flat as a left R-module proving that R is Von Neumann regular.

Remark : A copure-injective R-module need not be pure-injective.

Example—Let V be a countably infinite dimensional vector space over the field Q of rational numbers. Let R be the ring of linear operators of V. Then R is a Von Neumann regular ring. Let $\mathcal{M}$ be a maximal right ideal of R containing the two-sided ideal I of R of all elements of R of finite rank, Then, by Theorem 1 of Osofsky[15], $S = R/\mathcal{M}$ is not injective as an R-module. Let $x \in E(S) \setminus S$ and let A be a submodule of $E(S)$ maximal with respect to $S \subseteq A$ and $x \notin A$. Then $E(S)/A$ is subdirectly irreducible and hence, is cofinitely generated. So A is cofinitely related and by Proposition 3 (ii), A is copure-injective. Since A is not a direct summand of $E(S)$, A is not injective and so not pure-injective by Proposition 12.

From Propositions 9 and 11 we have :

Corollary 14—If R is commutative (co-) noetherian ring then R-module is pure-injective if and only if it is copure-injective.

We now recall the following Definitions.

(i) (Warfied[21], p. 707-708). Let I be any index set and let M be any R-module. Let M^I be the Ith cartesian power of M. For any finite subset I^* of I, and elements $r_i \in R$ $(i \in I^*)$, we define a group homomorphism (which will be a module homomorphism if R is commutive)

$\phi : M^I \to M$ by $\phi(x) = \sum_{i \in I} r_i x_i$. By a linear equation over R in M we mean a pair (ϕ, m) where ϕ is a homomorphism of the type defined above and $m \in M$. The solution set of this equation $S(\phi, m)$ is the set of all elements $x \in M^I$ such that $\phi(x) = m$.

A family of linear equations over R in M is said to be finitely soluble if the corresponding sets $S(\phi, m)$ have finite intersection property. An R-module M is said to be algebraically compact if every finitely soluble system of linear equations over R in M has a simultaneous solution.

(ii) An R-module M is said to be linearly compact (in the discrete topology) (Warfield[21], p. 711 and Vamos[20], p. 115) if every family of cosets in M with finite intersection property has nonempty intersection.

(iii) A commutative ring R is said to be classical (Vamos[20], p. 121) if $E(S)$ is linearly compact for every simple R-module S (or equivalently, every cofinitely generated R-module is linearly compact).

We now generalize the Proposition 11 to commutative classical rings by noting that commutative (co-) noetherian rings are classical (Vamos[19], Theorem 2 and Vamos[20], Proposition 4.1).

Proposition 15— Over a commutative classical ring R, every copure injective R-module is pure-injective.

PROOF : This follows from Proposition 9 of Warfield[21] that over a commutative ring R, every linearly compact R-module is algebraically compact and hence, pure-injective (Warfield[21], Theorem 2), from Theorem 8 and the facts that pure-injective R-modules are closed under taking arbitrary direct products and direct summands.

Corollary 16—For a commutative classical ring R, every pure shart exact sequence of R-modules is copure.

Corollary 17—For a commutative classical ring R, an R-module is pure-injective if and only if it is copure-injective.

PROOF : This follows from Propositions 9 and 15.

We next characterize the rings R for which every R-module is copure-injective.

From Theorem 8 we have :

Corollary 18—For a ring R the following conditions are equivalent.

(i) Every R-module is copure-injective.

(ii) Every R-module is a direct summand of a direct product of cofinitely related R-modules.

(iii) Every copure short exact sequence of R-modules splits.

Fieldhouse[5] (p. 15) calls a ring R a 'right PDS ring' if every pure submodule of an R-module is a direct summand (or, equivalently (Fieldhouse[5], Theorem 10.1) every pure short exact sequence of R-modules splits). Similarly we call a ring R a right CDS ring if R satisfies the equivalent conditions of Corollary 18.

Since a ring R is a right PDS ring if and only if every R-module is pure-injective, we have from Proposition 9 and Corollary 17 :

Proposition 19—(i) If R is a commutative PDS ring then R is a CDS ring,

(ii) If R is a commutative classical ring then R is a PDS ring if and only if it is a CDS ring.

We now have, from Proposition 19 and Theorem 10.3 of Fieldhouse[5] :

Corollary 20—Every commutative classical CDS ring is artinian.

We do not know whether a commutative artinian ring is CDS. However, we have from Theorem 10.4 of Fieldhouse[5] :

Corollary 21—Every commutative uniserial ring, that is, a commutative artinian principal ideal ring, is a CDS ring.

Remark 22 : We know that Dedekind domain need not be a CDS (= PDS) ring (cf. Proposition 19 (ii)) (e. g. Z, the ring of integers, is not a CDS ring since Z is not algebraically compact as an abelian group (Fuchs[6], Chapter VII, § 38, Exercise 1)). But every proper homomorphic image of a Dedekind domain, being an artinian principal ideal ring (and hence a uniserial ring), is a CDS ring by Corollary 21.

Remark 23 : We know (Cartan and Eilenberg[1], Proposition 6.1) that a ring R is right hereditary if and only if every homonorphic image of an injective R-module is injective. But neither a homomorphic image of a copure-injective R-module over a right hereditary ring need be copure-injective (see example (i) below) nor a ring R for which every every homomorphic image of a copure-injective R-module is copure-injective need be right hereditary (see example (ii)).

Examples—(i) We know that Z is a hereditary ring and copure-injective Z-modules are precisely the algebraically compact Z-modules (Theorem 2 of Warfield[21] and Corollary 14). Since there exists an algebraically compact abelian group with a homomorphic image which is not algebraically compact, it follows that there exists a copure-injective Z-module with a homomorphic image which is not copure-injective.
(ii) $R = Z/(4)$ is an artinian principal ideal ring and hence is a uniserial ring. Then it is a CDS ring by Corollary 21. So, every R-module, in particular every homomorphic image of a copure-injective R-module, is copure-injective. But R is not a hereditary ring since the maximal ideal $(2)/(4)$ of the local ring R is not projective as it is not free as an R-module.

Remark 24 : We know (Fuchs[6], §38, Exercise 3) that every pure homomorphic image of an algebraically compact abelian group is algebraically compact. But, in general, a copure homomorphic image of a copure-injective R-module need not be copure-injective.

Example—Let $R = \prod_{p \in P} Z/pZ$ where P is the set of all positive primes. Since R is self-injective (Faith[4], Exercise 5.64.2), it is self-copure-injective. Since R is not semisimple artinian, there exists, by the Theorem of Osofsky[14], an ideal I of R such that R/I is not injective as an R-module. Since R is a V-ring, R/I is not copure-injective as an R-module by Proposition 5.

We prove below, in Corollary 28, that every copure-homomorphic image of a copure-injective R-module over a Dedekind domain R is again copure-injective. Before proving this we take note of and derive a few results:

(i) Every finitely presented R-module is pure-projective.

(ii) Any R-module can be embedded in a pure short exact sequence $0 \to K \to P \to M \to 0$ of R-modules where P is pure-projective.

(iii) An R-module is pure-projective if and only if it is a direct summand of a direct sum of finitely presented R-modules.

Next we recall (Kaplansky[11], p. 332) that if R is a commutative integral domain, then an R-module M is said to be decomposable if M is a direct sum of cyclic R-modules and finitely generated R modules of rank one. Since over a Dedekind domain R, every finitely generated torsion-free R-module is projective (Cartan and Eilenberg[1], Chapter VII, Proposition 4.1 and a remark on p. 134), it follows that every decomposable R-module, over a Dedekind domain R, is a direct sum of cyclic R-modules and a projective R-module and hence it is pure-projective by (i) and (ii) of the results stated above.

Proposition 25—Over a Dedekind domain R every submodule of pure-projective R-module is pure-projective.

PROOF : We first observe that every pure-projective R-module is decomposable. This follows from : (a) Steinitz's theorem (Steinitz[17]) that every finitely generated R-module over a Dedekind domain R is decomposable; (b) the decomposability is a hereditary property for a Dedekind domain (Kaplansky[11], Theorem 4) and (c) form (iii) of the properties of pure-projective modules stated above.

From this observation and the observation made just before this proposition it follows that pure-projectivity and decomposability are equivalent for a Dedekind domain. The proposition now follows from Theorem 4 of Kaplansky[11].

Proposition 26—An R-module Q is pure-injective if and only if Q is injective relative to each pure short exact sequence $0 \to A \to B \to C \to 0$ of R-modules with B pure-projective.

PROOF : This follows by using arguments similar to the Proposition 5.2 of Chapter *I* of Cartan and Eilenberg[1] and the fact that the class of all pure short exact sequences of R-modules form a 'Proper Class' in the sence of MacLane[13] (p. 367) (Stenström[18], Propositions 2.2 and 9.1).

By Propositions 25 and 26 and using arguments similar to Theorem 5.4 of Chapter I of Cartan and Eilenberg[1], we have :

Proposition 27—Every pure homomorphic image of a pure-injective R-module over a Dedekind domain R is pure-injective.

Since, by Theorem 20 of Hiremath[8], purity and copurity are equivalent for a Dedekind domain we have, from Corollary 14 and Proposition 27 :

Corollary 28—For a Dedekind domain R, every copure homomorphic image of a copure-injective R-module is copure-injective.

Finally we prove the analogue of Schanuel's lemma for copure short exact sequences and copure-injective modules. We give the proof for the sake of completeness.

Proposition 22—Let A be an R-module and let $0 \to A \xrightarrow{f} Q \xrightarrow{g} B \to 0$ and $0 \to A \xrightarrow{f'} Q' \xrightarrow{g'} B' \to 0$ be copure short exact sequences of R-modules with Q, Q' copure-injective. Then $Q \oplus B'$ and $Q' \oplus B$ are isomorphic.

PROOF : Considering the usual pushout $C = (Q \oplus Q')/K$ where $K = \{(f(a), -f'(a)) : a \in A\}$ and the natural maps h, h' and by defining k, k' in the obvious way we obtain the following commutative diagram of exact rows and columns :

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & \xrightarrow{f} & Q & \xrightarrow{g} & B & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_B} & & \\
0 & \longrightarrow & Q' & \xrightarrow{h'} & C & \xrightarrow{k} & B & \longrightarrow & 0 \\
 & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle k'} & & & & \\
 & & B' & \xrightarrow{1_{B'}} & B' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

We prove that the second row and the second column are copure. Let M be any cofinitely related R-module and let $\alpha : Q' \to M$ be any homomorphism. Then by the copurity of the first row there is a $\beta : Q \to M$ such that $\beta f = \alpha f'$. Define $\phi : C \to M$ by $\phi((x, x') + K) = \beta(x) + \alpha(x')$ for $(x, x') + K$ in C. Clearly ϕ is a well-defined homomorphism and $\phi h' = \alpha$. So the second row is copure. Similarly the copurity of of the second column follows.

Now by the copure-injectivity of Q, Q' and the copurity of the second row and the second column, both the second row and the second column split by Proposition 3 (iii). Hence C is isomorphic to both $Q \oplus B'$ and $Q' \oplus B$.

ACKNOWLEDGEMENT

The author thanks Professor K. M. Rangaswamy for suggesting the problem, for his encouragement and for many useful discussions. The author also thanks the referee for his comments in the revision of this paper.

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