*Indian J. pure appl. Math.*, 20 (3) : 267-275, March 1989

# COMMENCEMENT OF COUETTE FLOW IN OLDROYD LIQUID WITH HEAT SOURCES

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(*Received* 11 *December* 1987; *after revision* 14 *June* 1988)

Commencement of Couette flow in Oldroyd liquid with heat sources has been analysed. Approximate solutions are derived by Galerkin technique. The effects of various parameters on the velocity field, the temperature field, the skin friction and the rate of heat transfer are discussed. It is observed that the flow is sensitive to the interactions of heat source and elasticity of the fluid.

## 1. INTRODUCTION

The problem of Couette flow between parallel walls has a rich stock of literature. Pai[1] has discussed the problem of unsteady Couette flow of a viscous inrompressible liquid between two plate walls occurring due to the sudden motion of one of the walls. The technique used for the solution of the equations is the principle of superposition and the solution occurs in the form of an infinite series of functions of the variables $Y$ and $t$. Nanda[2] has studied the same problem assuming the walls to be porous. Then, it has been extended by Katagiri[3] and Muhuri[4] Rath *et al.*[5] who have solved the heat transfer problem in case of unsteady Couette flow between two parallel walls having different temperature.

Unsteady Couette flow in non-Newtonian fluids has enormous applications in technology. Therefore, many researchers have obtained the approximate solutions for the unsteady Couette flow caused in visco-elastic fluids with different wall velocities under different physical situations. Generalized plane Couette flow of one Oldroyd fluid with either suction or injection at the stationary wall has been discussed by Mishra[6]. Soundalgekar[7] has analysed the problem of plane Couette flow of Walters liquid $B'$ with equal rate of injection at one wall and suction at the other (moving wall). Padhy[8] has analysed the commencement of unsteady Couette flow in second order liquid. The object of the present paper is to study the commencement of unsteady Couette flow in case of Oldroyd liquid contained between two porous walls in the presence of heat sources.

## 2. Basic Equations

The visco-elastic fluid model considered here is the simplified form of Oldroyd's $B$-liquid[9] whose stress-strain relation is given by

$$\left[1 + \lambda_1 \frac{\delta}{\delta t}\right] P'^{ik} = 2\eta_0 \left[1 + \lambda_2 \frac{\delta}{\delta t}\right] e^{ik}. \qquad \ldots(1)$$

In this equation $\lambda_1$ and $\lambda_2$ are the time of relaxation and retardation respectively, $\eta_0$ is the co-efficient of viscosity, $P'^{ik}$ and $e^{ik}$ are respectively the stress and the strain rate tensors, $\delta/\delta t$ denotes the convective derivative and, for a contravariant second order tensor, $b^{ik}$ is given by

$$\frac{\delta b^{ik}}{\delta t} = \frac{\partial b^{ik}}{\partial t} + v^m b^{ik}_{,m} + v^m_{,i} b^{mk} + v^m_{,k} b^{im}.$$

$v^i$ being the velocity vector and

$$2e_{ik} = v_{i,k} + v_{k,i}.$$

Neglecting the second order terms in $\lambda_1$ and $\lambda_2$ in eqn. (1) and following Gieskus[10], we get

$$P'^{ik} = 2\eta_0 e^{ik} - 2k_0 \frac{\delta}{\delta t} e^{ik} \qquad \ldots(2)$$

where

$$k_0 = \eta_0 (\lambda_1 - \lambda_2).$$

The momentum equations are given by

$$\rho \left[\frac{\partial v^i}{\partial t} + v^j v^i_{,j}\right] = - p_{,i} + p^{ij}_{,j} \qquad \ldots(3)$$

where $\rho$ is the density of the medium and $p$ an arbitrary isotropic pressure. The equation of continuity is

$$U^i_{,i} = 0 \qquad \ldots(4)$$

and the energy equation is given by

$$\rho C_p \frac{DT}{Dt} = k^* \nabla^2 T + \Phi \qquad \ldots(5)$$

where $C_p$ is the specific heat at constant pressure, $k^*$ the thermal conductivity and $\Phi$ the dissipation function prescribed by

$$\Phi = e_{ij} P_{ij}. \qquad \ldots(6)$$

## 3. Formulation of the Problem

Here, the unsteady Couette flow of an incompressible viscoelastic fluid begins with the sudden motion of the lower wall with time varying velocity $At'^n$ where $n$ is positive. The $x'$-axis is chosen along the lower wall and the $y'$-axis normal to it. The upper plane is specified by the equation $y' = L$ where the number $L$ will be defined later. It is also supposed that the walls extend to infinity in both sides of the $x'$-axis. We consider the suction and injection velocity $V$ at the walls to be constant. Then the velocity components $u'$ and $v'$ at any point $(x', y')$ in the flow field compatible with the equation of continuity are given by

$$\left.\begin{aligned} u' &= u'(y, t) \\ v' &= V. \end{aligned}\right\} \qquad \ldots(7)$$

Under these conditions and also due to the shearing action of the fluid layers, the flow field and the temperature field with viscosity dissipation for an Oldroyd fluid are characterized by the following equations :

$$\rho\left(\frac{\partial u'}{\partial t'} + V\frac{\partial u'}{\partial y'}\right) = \eta_0 \frac{\partial^2 u'}{\partial y'^2} - k_0\left(\frac{\partial^3 u'}{\partial y'^2 \partial t'} + V\frac{\partial^3 u'}{\partial y'^3}\right) \qquad \ldots(8)$$

$$\rho C_p\left(\frac{\partial \theta'}{\partial t'} + V\frac{\partial \theta'}{\partial y'}\right) = K^* \frac{\partial^2 \theta'}{\partial y'^2} + \eta_0\left(\frac{\partial u'}{\partial y'}\right)^2 - k_0\left[\frac{\partial^2 u'}{\partial y'\,\partial t'}\cdot\frac{\partial u'}{\partial y'} + V\frac{\partial u'}{\partial y'}\cdot\frac{\partial^2 u'}{\partial y'^2}\right] + S'(\theta' - \theta_L). \qquad \ldots(9)$$

The notations $\rho$, $C_p$, $\eta_0$, $K_0$, $\theta'$ and $K^*$ used in the above equations are respectively the density, specific heat, co-efficient of viscosity, coefficient of elasticity, temperature at any point and thermal conductivity of the fluid. The last term in eqn. (9) represents the source-sink term with $S'$ as its strength.

The adequate boundary conditions are

$$\left.\begin{aligned} &t' = 0 : u' = 0 \text{ for all } y' \\ &t' > 0 : \left.\begin{aligned} u' &= At'^n \text{ for } y' = 0 \\ u' &= 0 \quad \text{for } y' = L \end{aligned}\right\} \end{aligned}\right\} \qquad \ldots(10)$$

and

$$\left.\begin{aligned} &t' = 0 : \theta' = 0 \quad \text{for all } y' \\ &t' > 0 : \left.\begin{aligned} \frac{\partial \theta'}{\partial y'} &= 0 \text{ for } y' = 0 \\ \theta' &= \theta_L \text{ for } y' = L. \end{aligned}\right\} \end{aligned}\right\} \qquad \ldots(11)$$

The fact $\frac{\partial \theta'}{\partial y'} = 0$ when $y' = 0$ implies that the lower wall is a non conducting one. Now, we introduce the following dimensionless variables and parameters :

$$\left.\begin{aligned} y &= \frac{y'}{\sqrt{\nu_1 T}}, \; t = \frac{t'}{T}, \; u = \frac{u'}{AT^n} \\ R &= \frac{V\sqrt{T}}{\sqrt{\nu_1}}, \; R_c = \frac{\lambda_1 - \lambda_2}{T}, \; \sigma = \frac{\nu_1 \rho C_p}{K^*} \\ E &= \frac{A^2 T^{2n}}{C_p \theta_L}, \; \theta = \frac{\theta' - \theta_L}{}, \; S = \frac{4S' \nu_1}{V^2} \\ \nu_1 &= \frac{\eta_0}{\rho} \text{ and } L = \sqrt{\nu_1 T} \end{aligned}\right\} \quad \ldots(12)$$

where $T$ is some reference time, $\theta_L$ the temperature of the upper plate, and $R$, $R_c$ $E$, $\sigma$ and $S$ are respectively the suction parameter, elastic parameter, Eckert number, Prandtl number non-dimensional source-parameter.

By the use of (12), eqns. (8) and (9) are reduced to non-dimensional form as follows :

$$\frac{\partial u}{\partial t} + R\frac{\partial^2 u}{\partial y} - \frac{\partial^2 u}{\partial y^2} + RR_c\frac{\partial^3 u}{\partial y^3} + R_c\frac{\partial^3 u}{\partial y^2\,\partial t} = 0 \quad \ldots(13)$$

$$\frac{\partial \theta}{\partial t} + R\frac{\partial \theta}{\partial y} - \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial y^2} + R_c\,E\,\sigma\left[R\,\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial y\,\partial t}\right]$$

$$+ \sigma E\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{4}R^2 S\theta = 0. \quad \ldots(14)$$

The modified boundary conditions are

$$\left.\begin{aligned} t = 0 &: u = 0 \text{ for all } y \\ t > 0 &: \left.\begin{aligned} u &= t^n \text{ for } y = 0 \\ u &= 0 \text{ for } y = 1 \end{aligned}\right\} \end{aligned}\right\} \quad \ldots(15)$$

and

$$\left.\begin{aligned} t = 0 &: \theta = 0 \text{ for all } y \\ t > 0 &: \begin{aligned} \frac{\partial \theta}{\partial y} &= 0 \text{ for } y = 0 \\ \theta &= 0 \text{ for } y = 1. \end{aligned} \end{aligned}\right\} \quad \ldots(16)$$

## 3. Method of Solution

It is difficult to solve equations (13) and (14) which are of order 3. However, one of the possibility is to derive one solution for small values of $R_c$ ($< 1$) and then we assume

$$u = u_0 + R_c\,u_1 + O\,(R_c^2). \quad \ldots(17)$$

Substituting (17) in (13) and equating the like powers of $R_c$ we get

$$\frac{\partial u_0}{\partial t} + R \frac{\partial u_0}{\partial y} - \frac{\partial^2 u_0}{\partial y^2} = 0 \qquad ...(18)$$

$$\frac{\partial u_1}{\partial t} + R \left(\frac{\partial u_1}{\partial y^3} + \frac{\partial^3 u_0}{\partial y^3}\right) - \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^3 u_0}{\partial y^2 \partial t} = 0. \qquad ...(19)$$

The boundary conditions for eqns. (18) and (19) are obtained from eqns. (10) and (17) as follows:

$$\left.\begin{array}{l} t = 0 : u_0 = 0,\ u_1 = 0 \ \forall\ y \\ t > 0 : \left.\begin{array}{l} u_0 = t^n,\ u_1 = 0 \text{ for } y = 0 \\ u_0 = 0\ u_1 = 0 \text{ for } y = 1 \end{array}\right\} \end{array}\right\}. \qquad ...(20)$$

Equations (18), (19) and (14) subject to the conditions (20) and (11) are solved using Galarkin technique. For this the following infinite forms for $u_0$, $u_1$ and $\theta$ are proposed which satisfy the initial and boundary conditions.

$$u_0 \approx t^n (1 - y) + a_1\, ty (1 - y) + a_2\, t^2 y^2 (1 - y^2) + \ldots$$
$$a_3\, t^3 y^3 (1 - y)^3 + \ldots \qquad ...(21)$$

$$u_1 \approx b_1\, ty (1 - y) + b_2\, t^2 y^2 (1 - y)^2 + b_3\, t^3 y^3 (1 - y)^3 + \ldots \qquad ...(22)$$

$$\theta \approx c_1\, t (1 - y^2) + c_2\, t^2 y (1 - y^2)^2 + c_3\, t^3 y^2 (1 - y^2)^3 + \ldots \qquad ...(23)$$

where $a_i$ $b_i$ and $c_i$ ($i = 1, 2, 3$) are arbitrary constants to be determined. Substituting (21) – (23) into eqns. (18), (19) and (14) and neglectings $a_i$, $b_i$ and $c_i$ for $i \geqslant 4$, the defect functions $D_{u_0}$, $D_{u_1}$ and $D_\theta$ are obtained which are minimized by the technique of orthogonalisation and result in the following nine double integrals

$$\left.\begin{array}{l} \int_0^1 \int_0^1 D_{u_0}\, t^j y^j (1 - y)^j\, dt\, dy = 0 \\ \int_0^1 \int_0^1 D_{u_1}\, t^j y^j (1 - y)^j\, dt\, dy = 0, \\ \int_0^1 \int_0^1 D_\theta\, t^j j^{j-1} (1 - y^2)^j\, dt\, dy = 0. \end{array}\right\} j = 1, 2, 3 \qquad ...(24)$$

It is note worthy here that $t \in [0, 1]$.

Performing the integrations, we arrive at the nine algebraic equations involving the parameters $a_j$, $b_j$ and $c_j$ ($j = 1, 2, 3$). These nine linear equations are solved which give the constants $a_j$, $b_j$ and $c_j$ and hence the velocity field $u = u_0 + R_c\, u_1$ and the temperature field $\theta$.

The non-dimensional shear stress $\tau_{xy}$ is given by

$$\tau_{xy} = \frac{P_{xy} \sqrt{T}}{\rho \sqrt{\nu_1}} AT^n = \frac{\partial u}{\partial y} - R_c \left\{R \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y\, \partial t}\right\}. \qquad ...(25)$$

The shear stresses at the plates $\tau_0 = \tau_{xy} \mid_{y=0}$ and $\tau_1 = \tau_{xy} \mid_{y=1}$ are calculated. The rate of heat transfer at the plates

$$Nu_0 = - \left.\frac{\partial \theta}{\partial y}\right|_{y=0} \text{ and } Nu_1 = - \left.\frac{\partial \theta}{\partial y}\right|_{y=1} \text{ are also calculated.}$$

## 4. Results and Discussions

The flow characteristics are studied for two positive values of $n$, i. e.,

(i) $n = 1$, constant acceleration

(ii) $n = \frac{1}{2}$, variable acceleration.

It is observed from the Fig. 1 that the flow field exhibts almost a diametrically opposite behaviour for values of $R_c$ greater than zero. It is further observed that the velocity

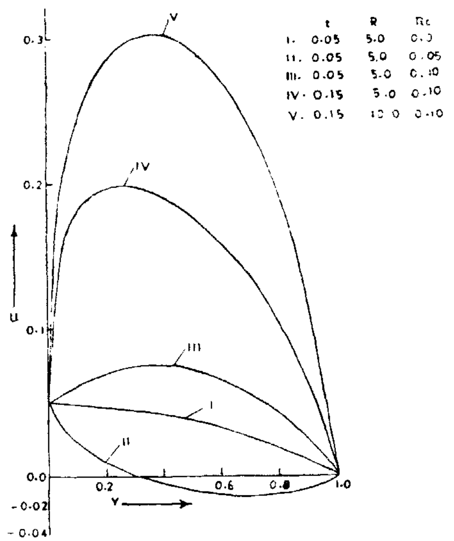


Fig. 1. Effect of $R$ and $R_c$ on velocity field, $n = 1.0$

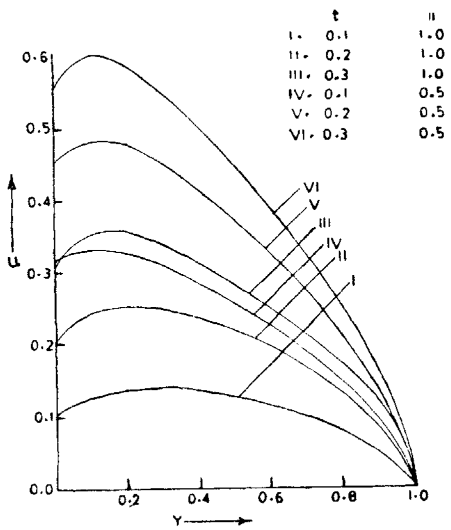


Fig. 2. Effect of $t$ and $n$ on velocity field, $R = 5.0$, $R_c = 0.1$

rises sharply with increasing values of $t$ and this rise is quantitatively presented as 152%. To record the relative influence of the two parameters $t$ and $n$ on the flow pattern, it can be said from Fig. 2 that time plays a more effective role in increasing the velocity than $n$.

It is noticed from Fig. 3 that an increase in $R_c$ is to decrease the temperature at all points and also an increase in time decreases it further.

Figure 4 shows that an increase in $R$ is to increase the temperature at all points.

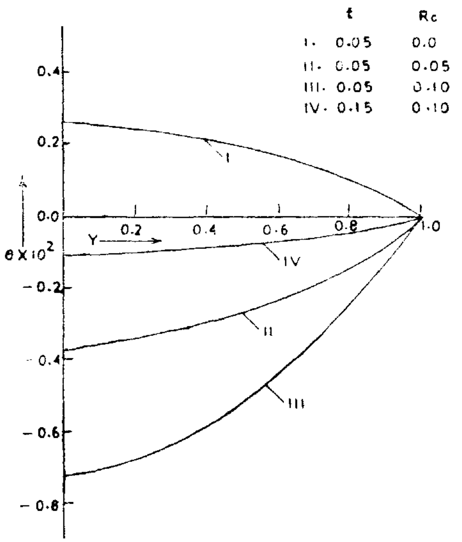


FIG. 3. Effect of $R_c$ on temperature field, $R = 5.0$, $n = 1.0$, $S = 0.1$, $\sigma = 0.0$, $E = 0.01$

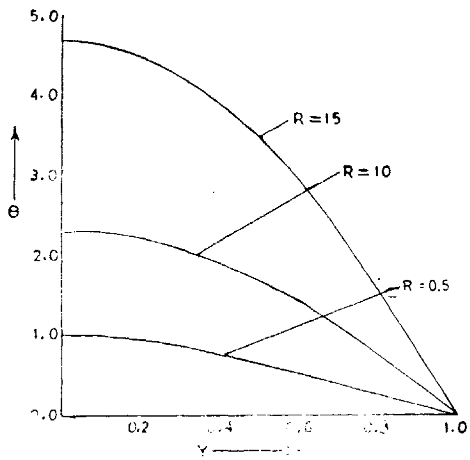


FIG. 4. Effect of $R$ on temperature field, $R_c = 0.05$, $n = 1.0$, $S = 0.1$ $E = 0.001$, $\sigma = 0.5$

Figure 5 shows that the temperature field assumes negative values for $S = 1.0$ and $S = 0.5$. Further, it is observed that the temperature decreases with the decrease of source strength from 1.0 to 0.5 but for $S = 0.1$ it increases sharply.

Figure 6 shows that the shearing stress at the lower wall decreases with an increase in time for both variable and constant acceleration. It is also noticed that $\tau_0$, in case of constant acceleration is greater than the variable acceleration for fixed

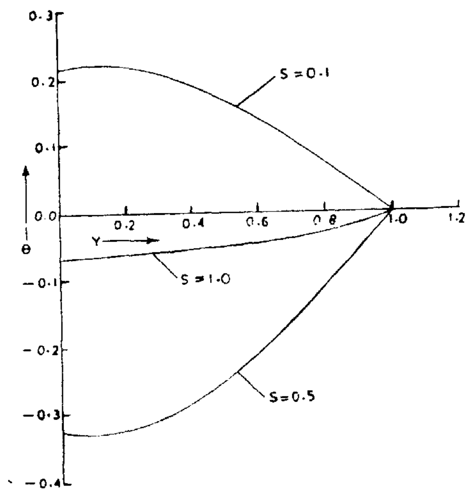


FIG. 5. Effect of $S$ on temperature field, $R = 10.0$, $t = 0.05$, $R_c = 0.1$, $n = 1.0$, $\sigma = 0.2$, $E = 0.01$

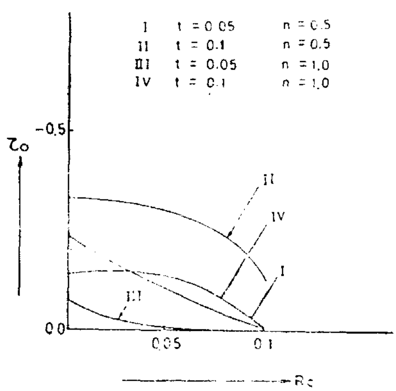


FIG. 6. Shearing stress at the lower wall, $R = 0.1$

value of $t$. Whereas the shearing stress at the upper wall $\tau_1$ depicts an opposite behaviour for both $t$ and $n$ (Fig. 7). Further it is observed that the elasticity of the liquid increases both $\tau_0$ and $\tau_1$.

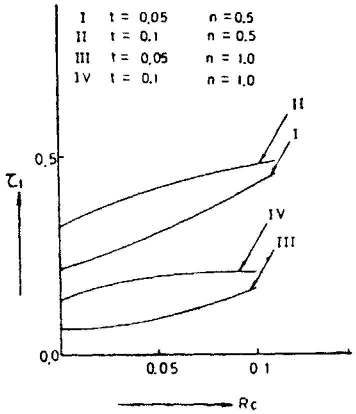


FIG. 7. Shearing stress at the upper wall, $R = 0.5$.

TABLE I

*Values of the rates of heat transfer, $Nu_0$ and $Nu_1$ $S = 0.1$, $\sigma = 0.1$, $E = 0.01$*

| | | $n$ | 0.5 | | 1.0 | |
|---|---|---|---|---|---|---|
| $R$ | $t$ | $R_c/n$ | $Nu_0$ | $Nu_1$ | $Nu_0$ | $Nu_1$ |
| | 0.05 | 0.0 | 0.00004 | 0.00160 | 0.00003 | 0.00131 |
| | | 0.05 | 0.00006 | 0.00194 | 0.00006 | 0.00193 |
| | | 0.10 | 0.00004 | 0.00159 | 0.00003 | 0.00129 |
| 0.5 | | | | | | |
| | | 0.0 | 0.00019 | 0.00321 | 0.00375 | 0.00262 |
| | 0.10 | 0.05 | 0.00024 | 0.00389 | 0.00024 | 0.00387 |
| | | 0.10 | 0.00019 | 0.00319 | 0.00015 | 0.00258 |
| | | 0.0 | 0.00015 | 0.00628 | 0.00012 | 0.00520 |
| 5.0 | 0.05 | 0.05 | −0.00112 | −0.01647 | −0.00095 | −0.01449 |
| | | 0.10 | −0.00047 | −0.01097 | −0.00032 | −0.00737 |

TABLE II

*Showing the effect of $S$ on the rates of heat transfer, $Nu_0$ and $Nu_1$ $t = 0.05$, $R_c = 0.1$, $n = 1.0$, $\sigma = 0.2$ $E = 0.01$*

| $R$ | $S$ | $Nu_0$ | $Nu_1$ |
|---|---|---|---|
| | 0.1 | 0.00004 | 0.00140 |
| 0.5 | 0.5 | 0.00004 | 0.00140 |
| | 1.0 | 0.00004 | 0.00141 |
| | 0.1 | 0.00014 | 0.43116 |
| 10.5 | 0.5 | 0.00735 | −0.65259 |
| | 1.0 | 0.00496 | −0.13865 |

Table I describes the effects of $R$, $R_c$, $t$ and $n$ on the Nusselt number. $Nu_0$ and $Nu_1$ both increase when $R_c$ takes the values from 0.0 to 0.05. The rate of heat transfer falls for $R_c = 0.1$. From Table II it is ascertained that the increase in source strength maintains almost a constant rate of heat transfer at both the walls for $R = 0.5$.

## Acknowledgement

The authors acknowledge their thanks to the referee for the valuable comments and Professor B. P. Acharya for suggestions to improve the paper.

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