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ON RAYLEIGH WAVES IN GREEN-LINDSAY'S MODEL OF GENERALIZED THERMOELASTIC MEDIA

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Rayleigh wave formulation in a half-space exhibiting generalized thermoelasticity propeities of Green-Lindsay model has been made. The case of Lord and Shulman's generalized thermoelasticity is shown to occur as a special case. Approximations to the frequency equation have been made for different ranges of the parameters involved. In the case of small coupling, a numerical study has been made to show the effect of the relaxation parameters on the amplitude attenuation factor and also on the velocity of propagation

INTRODUCTION

Thermo-elastic Rayleigh waves in semi-infinite isotropic solids have been studied by Deresiewicz[1], Lockett[2], Chadwick and Sneddon[3], Chadwick[4]. Chakraborty and Pal[5] extended the problem to transversely isotropic medium. These surface waves are found to propagate with a speed in general dependent on the thermal properties as well as on the wave-length of the waves. In the case of small reduced frequency, however, the thermal effect on the wave speed is found to be negligible.

The physical foundations of the above problems were the formulations of Biot[6] and Lessen[7]. The coupled heat conduction equations used by them permitted infinite speeds of propagation of thermal signals. Lord and Shulman[8], proposed generalized thermo-elasticity equations, in which they modified Fourier's heat conduction equation by taking into account the time needed for acceleration of the heat flow. This leads to changes in the energy equation of a coupled theory of theoremo-elasticity by the incorporation of a relaxation term. Green and Lindsay[9] proposed generalized constitutive equations starting from the thermo-dynamical principles without altering Fourier's heat conduction equation. The Green-Lindsay thermoelasticity equations involve two new parameters. Both the Lord-Shulman and Green-Lindsay formulations give finite speeds of thermal wave propagation and also a "second sound effect". However, Green-Lindsay formulation appears to be theoretically more satisfying and Lord-Shulman case follows as a special case of Green-Lindsay in many problems. In the linearized case, according to the Green-Lindsay theory, the relaxation parameters a and a^* in the stress-strain temperature relation and the equation of heat conduction respectively are independent but the inequality $a \geqslant a^* \geqslant 0$ is to be satisfied for uniqueness of temperature and the speed of heat conduction which depends on a^* is finite only if the stresses depend on the time rate of temperature.

Nayfeh and Nemat-Nasser[11] used the Lord-Shulman theory to study plane thermoelastic surface waves in a half space. On the basis of Green-Lindsay model, Agarwal[12] studied plane thermo-elastic waves—their propagation and stability. The two results in general are different. They coincide however, in the special case when the two relaxation parameters are equal.

The present paper is concerned with the problem of thermoelastic Rayleigh wave in semi-infinite solids of Green-Lindsay's model. It is found that the relaxation parameters a, a^* for Green-Lindsay theory contribute to the frequency equation terms of order higher than $\chi^{-1/2}$, χ being the reduced frequency. It is seen that when the two relaxation parameters are equal, the results reduce to the case studied by Nayfeh and Nemat-Nasser[11] based on Lord-Shulman's theory.

Approximations to the frequency equations for different ranges of the parameters involved have been obtained. A numerical study of the change of the amplitude attenuation factor and also of velocity of propagation with the relaxation parameter a have been made in the case of small thermo-elastic coupling.

Basic Equations

The equations governing linear thermoelastic interactions in a homogeneous and isotropic solid free from body force and heat sources, as proposed by Green and Lindsay[9] and Green[10] are :

(a) the strain-displacement relations :

$$2e_{ij} = u_{i,j} + u_{j,i} \qquad \ldots(1)$$

(b) the stress-strain temperature relations :

$$\tau_{ij} = \lambda \Delta \delta_{ij} + 2 \mu e_{ij} - \gamma (T + a\dot{T}) \delta_{ij} \qquad \ldots(2)$$

(c) the equation of motion :

$$\mu \nabla^2 \vec{u} + (\lambda + \mu) \nabla \operatorname{div} \vec{u} - \gamma \nabla (T + a \dot{T}) = \rho \ddot{\vec{u}} \qquad \ldots(3)$$

(d) the equation of heat conduction :

$$\kappa \nabla^2 T - \rho c_v (\dot{T} + a^* \ddot{T}) - \gamma T_0 \operatorname{div} \dot{\vec{u}} = 0 \qquad \ldots(4)$$

where

$$a \geqslant a^* \geqslant 0$$

e_{ij} = cartesian components of the linear strain-tensor

τ_{ij} = cartesian components of the linear stress-tensor

δ_{ij} = Kronecker's delta

$\Delta = u_{i,j}$ = dilatation,

$\gamma = (3\lambda + 2\mu)\,\alpha_t$; λ, μ = Lame's constant

α_t = coefficient of linear thermal expansion

κ = coefficient of thermal conductivity,

ρ = constant mass density

c_v = specific heat at constant volume

T = the change in the absolute basic temperature T_0.

a, a^* = thermal relaxation times (constitutive coefficients).

Formulation of the Problem

For Rayleigh type waves in the half space $z \geqslant 0$, using the representation of displacement components :

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z},\; u_y = 0,\; u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \qquad \ldots(5)$$

where ϕ and ψ are functions of x, z and t, eqns. (3) and (4) are satisfied if

$$\frac{\partial^2 \phi}{\partial t^2} = c_1^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - \frac{\gamma}{\rho}\,(T + a\,\dot{T}) \qquad \ldots(6)$$

$$\frac{\partial^2 \psi}{\partial t^2} = c_2^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \qquad \ldots(7)$$

and

$$\rho\, c_v\,(T + a^*\,\ddot{T}) + \gamma\, T_0\, \frac{\partial}{\partial t} \left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right\} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \qquad \ldots(8)$$

with $c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$.

Using the quantities

$$x' = x/(c_1/\omega^*),\; z' = z/(c_1/\omega^*),\; t' = t\,\omega^*,$$

$$u'_x = \frac{u_x}{(c_1/\omega^*)},\; u'_z = \frac{u_z}{(c_1/\omega^*)},\; T' = \frac{\gamma T}{\rho c_1^2}$$

$$\phi' = \frac{\phi}{(c_1/\omega^*)^2},\; \psi' = \frac{\psi}{c_1/\omega^*)^2},\; a' = a\omega^*,\; a^{*\prime} = a^*\,\omega^*$$

where $\omega^* = \rho\, c_v\, c_1^2 / \kappa$, in eqns. (6), (7), (8) and suppressing the primes, were obtain the equations in dimensionless form :

$$u_x = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z},\ u_z = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \qquad ...(9)$$

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) - (T + a\dot{T}) \qquad ...(10)$$

$$(\dot{T} + a^* \ddot{T}) + \epsilon \frac{\partial}{\partial t}\left\{\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right)\right\} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \qquad ...(11)$$

and

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{\nu^2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \qquad ...(12)$$

where the thermo-elastic coupling is given by

$$\epsilon = \frac{\gamma^2 T_0}{\rho^2 c_v c_1^2} \text{ and } \nu^2 = \frac{c_1^2}{c_2^2}. \qquad ...(13)$$

Stress and thermal conditions of the problem on the boundary $z = 0$ are $\tau_{zx} = \tau_{zz} = 0$ and $\frac{\partial T}{\partial z} + hT = 0.$ ...(14)

SOLUTION

For thermoelastic surface waves in the half-space propagating in x-direction the functions $\{T, \phi, \psi\}$ may be taken in the form

$$\{T, \phi, \psi\} = \{\hat{T}(z), \hat{\phi}(z), \hat{\psi}(z)\} \exp i(\eta x - \chi t). \qquad ...(15)$$

Substituting (15) in (9) – (11) and remembering that $\hat{T}, \hat{\phi}, \hat{\psi} \to 0$ as $z \to \infty$ for surface waves, the solution is obtained as

$$\phi = [A \exp(-\eta \beta_1 z) + B \exp(-\eta \beta_2 z)]. \exp i(\eta x - \chi t) \qquad ...(16)$$

$$\psi = C. \exp[-\eta \beta_3 z + i(\eta x - \chi t)] \qquad ...(17)$$

and

$$T = \frac{1}{1 - ia\chi} [A \{\chi^2 + \eta^2 (\beta_1^2 - 1)\} \exp(-\eta \beta_1 z)$$

$$+ B \{\chi^2 + \eta^2 (\beta_2^2 - 1)\} \exp(-\eta \beta_2 z)]. \exp i(\eta x - \chi t) \qquad ...(18)$$

where $\beta_3^2 = (1 - c^2 \nu^2), c^2 = x^2/\eta^2$ and β_1, β_2 are those roots of the following equation for which Re $(\beta) > 0$:

$$\beta^4 + \beta^2 [-2 + c^2 \{(1 + a^* + \epsilon a) + i(1 + \epsilon)/\chi\}] + 1 - c^2 \{(1 + a^*$$

$$+ \epsilon a) + \frac{i}{x}(1 + \epsilon)\} + c^4 (a^* + \frac{i}{x}) = 0. \qquad ...(19)$$

On using the boundary conditions (14), $A : B : C$ are eliminated giving rise to the frequency equation

$$(2 - \nu^2 c^2)^2 (\beta_1^2 + \beta_2^2 + \beta_1 \beta_2 - 1 + c^2) - 4 \beta_1 \beta_2 \beta_3 (\beta_1 + \beta_2)$$

$$= - \frac{h}{\eta} \{(\beta_1 + \beta_2)(2 - \nu^2 c^2)^2 - 4 \beta_3 (\beta_1 \beta_2 + 1 - c^2)\}. \quad \ldots(20)$$

DISCUSSION

Equation (20) along with (19) determines the velocity of thermoelastic Rayleigh waves in a relaxing medium. In order to have an idea of the effect of the relaxation parameters on the velocity of propagation we shall study below in detail the special case of small thermoelastic coupling :

Case (a): Small Thermal Coupling

For most of the materials ϵ is small at normal temperature. Hence we may make an approximation of the frequency equation assuming $\epsilon \ll 1$.

For $\epsilon \ll 1$, we get from (19)

$$\beta_1 \simeq (1 - c^2)^{1/2} \left[1 - \frac{\epsilon}{2} \frac{c^2\left(a + \frac{i}{\chi}\right)}{(1 - a^* - \frac{i}{\chi})(1 - c^2)} \right. \quad \ldots(21)$$

$$\beta_2 \simeq \{1 - c^2 (a^* + \frac{i}{\chi})\}^{1/2} [1 + \frac{\epsilon}{2} \frac{c^2 (a^* + \frac{i}{x})(a + \frac{i}{x})}{(1 - a^* - i/\chi)\{1 - c^2 (a^* + I/\chi} \Big] .$$

The frequency equation (20) with $h = 0$ thus reduces to an equation in which the parameters a, a^* are involved :

$$(2 - \nu^2 c^2)^2 [1 - c^2 \{(a^* + \epsilon a) + \frac{i}{\chi}(1 + \epsilon)\} + (1 - c^2)^{1/2} \{1 - c^2 (a^*$$

$$+ \frac{i}{\chi})\}^{1/2} (1 - \frac{\epsilon}{2} \frac{c^2 (a + i/\chi)}{(1 - c^2)\{1 - c^2 (a^* + i/\chi)\}})] - 4 (1 - c^2 \nu^2)^{1/2}$$

$$(1 - c^2)^{1/2} \{1 - c^2 (a^* + \frac{i}{x})\}^{1/2} [1 - \frac{\epsilon}{2}] \frac{c^2 (a + i/\chi)}{(1 - c^2)\{1 - c^2 (a^* + i/\chi)\}]}$$

$$[(1 - c^2)^{1/2} \{1 - \epsilon/2 \frac{c^2 (a + i/\chi)}{(1 - a^* - i/\chi)(1 - c^2)}\} + \{1 - c^2 (a^* + i/\chi)\}^{1/2}$$

$$\{1 + \epsilon/2 \frac{c^2 a^* + i/\chi)(a + i/\chi)}{(1 - a^* - i'\chi)\{1 - c^2 (a^* + i/\chi)\}}\}] = 0. \quad \ldots(22)$$

If we put

$$c^2 = c^{*2} + \epsilon\,(\xi_1 + i\xi_2)$$

where c^* is the classical Rayleigh wave velocity and ξ_1 and ξ_2 are two reals depending on the reduced frequency χ and a, a^*,

then

$$\eta = \frac{\chi}{c^*}\left(1 - \frac{\epsilon\,\xi_1}{2c^{*2}} - \frac{i\,\epsilon\,\xi_2}{2\,c^{*2}}\right)$$

The velocity of propagation being $\left(c^* + \frac{\epsilon\,\xi_1}{2\,c^*}\right)$ the waves exhibit dispersion and the amplitude-attenuation factor $= \exp\left[\frac{\epsilon\,\chi\,\xi_2\,x}{2c^{*3}}\right]$ with $\xi_2 < 0$.

Substituting the above value of c^2 in (22) and solving numerically for different values of the parameters we have calculated the velocity of propagation and the amplitude-attenuation factor. In Fig. 1, we have plotted velocity of propagation (χ) 10^3 against the relaxation parameter a for $\epsilon = 0.05$, $(\chi) = 0.1$ and $a^* = 0.2$ and in Fig. 2 amplitude attenuation factor (χ) 10^3 has been plotted against the relaxation parameter a for the same value of ϵ, χ and a^* for the case $x = 1$.

From Fig. 1, it is seen that the velocity of propagation decreases gradually as the relaxation parameter a increases, though the change is very small. This may be interpreted as solving down of Rayleigh waves with increasing heat conductivity (Nayfeh and Nemat-Nasser[11], p. 55).

Similarly, in the Fig. 2, it is seen that amplitude-attenuation factor decreases with increase in the relaxation parameter a.

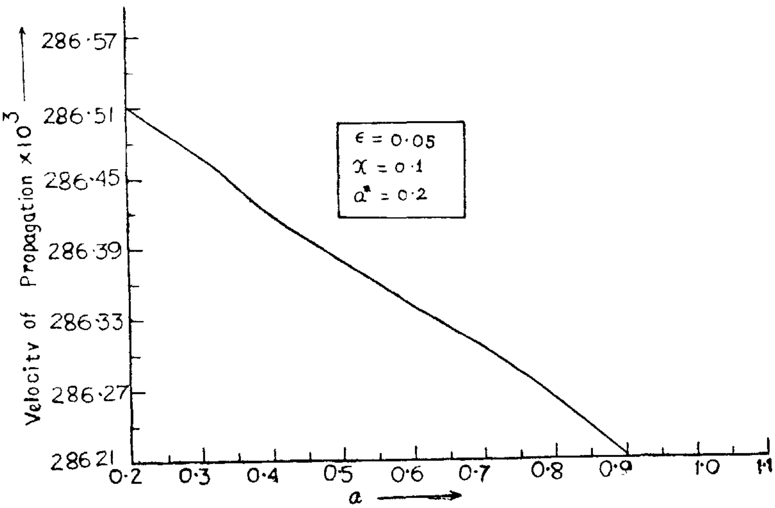

FIG. 1.

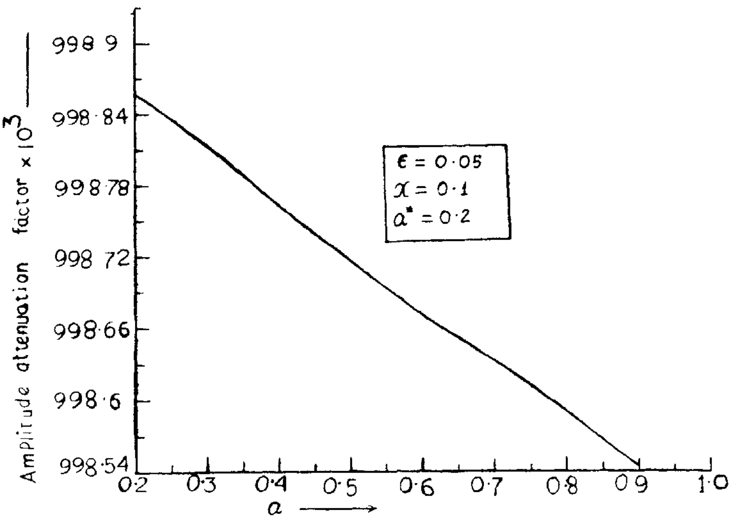

FIG. 2.

Case (b) : Small reduced frequency $\chi \ll 1$.

It is seen that the characteristic frequency ω^* is much greater than the frequencies attainable even in experiments employing ultrasonic pulses. Thus we may assume $\omega \ll \omega^*$, i. e. $\chi \ll 1$. Expanding the terms of the frequency equation in powers of the reduced frequency χ, the frequency equation (20) can be written as

$$
\begin{aligned}
&(2 - \nu^2 c^2)^2 \Big[- \frac{i\, c^2}{\chi} (1 + \epsilon) + \frac{(1 - i)\, c}{\sqrt{2}.\, \chi^{1/2}} (1 + \epsilon - c)^{1/2} + 1 - c^2 \\
&(a^* + \epsilon a) + \frac{(1 + i)}{2\sqrt{2}} \frac{\{1 - c^2 (1 + a^* + \epsilon a) + a^* c^4\}}{c\,(1 + \epsilon - c^2)^{1/2}} .\, \chi^{1/2} \Big] \\
&\quad - 4\,(1 - \nu^2 c^2)^{1/2} \Big[- \frac{i\, c^2}{\chi} (1 + \epsilon)^{1/2} (1 + \epsilon - c^2)^{1/2} \\
&\quad + \frac{(1 + i)}{\sqrt{2}} \frac{c}{\chi^{1/2}} . \frac{(1 + \epsilon - c^2)}{(1 + \epsilon)^{1/2}} + (1 + \epsilon) \\
&\quad \times \frac{\{3 - 2c^2 (1 + a^* + \epsilon a) + a^* c^4\} - c^2 \{2 - c^2 (1 + a^* + \epsilon a)\}}{2\,(1 + \epsilon)^{1/2} (1 + \epsilon - c^2)^{1/2}} \\
&\quad + \frac{(1 + i)}{2\sqrt{2}} . \frac{\chi^{1/2}}{c} \frac{\{1 - c^2 (1 + a^* + \epsilon a) + a^* c^4\}}{(1 + \epsilon)^{1/2}} \Big] \\
&= - \frac{h}{\eta} \Big[\Big\{ \frac{1 - i}{\sqrt{2}} . \frac{c}{\chi^{1/2}} . (1 + \epsilon)^{1/2} + \frac{(1 + \epsilon - c^2)^{1/2}}{(1 + \epsilon)^{1/2}} + \frac{(1 + i)}{2\sqrt{2}} \\
&\quad \times \frac{\chi^{1/2}}{c} . \frac{2 - c^2 (1 + a^* + \epsilon\, a)}{(1 + \epsilon)^{1/2}} \Big\} (2 - \nu^2 c^2)^2 - 4\,(1 - \nu^2 c^2)^{1/2}
\end{aligned}
$$

(equation continued on p. 283)

$$\times \left\{ \frac{1-i}{\sqrt{2}} \cdot \frac{c}{\chi^{1/2}} \cdot (1+\epsilon-c^2)^{1/2} + \frac{1+i}{2\sqrt{2}} \frac{\chi^{1/2}}{c} \right.$$

$$\left. \times \frac{1-c^2(1+a^*+\epsilon a)+a^* c^4}{(1-\epsilon-c^2)^{1/2}} + 1 - c^2 \right\}]. \qquad \ldots(24)$$

From (24), we therefore find that the relaxation parameter's 'a', a^* are involved only in the terms of order higher than $\chi^{-1/2}$. Hence the effect of these parameters is negligible for small χ.

The results correspondıng to zero flux of heat across the boundary may be obtained by putting $h = 0$. Therefore putting $h = 0$ in the frequency equation (24) and retaining terms only $O(\chi^{-1})$ we get

$$(2 - \nu^2 c^2)^2 = 4(1 - \nu^2 c^2)^{1/2} . \frac{1-\epsilon-c^2}{(1+\epsilon)^{1/2}} . \qquad \ldots(25)$$

Equation (25), is identital with the corresponding equation in coupled thermoelastic medium[2].

Case (c) : Lord-Shulman Case

It is to be noted that eliminating the temperature from the displacement equation (3) and the heat conduction equation (4) and then putting $a = a^*$ results in an equation for displacement which is the same as in Lord-Shulman's case with 'a' as the relaxation parameter. Therefore in Green-Lindsay's theory, all wave problems and source problems reduce to Lord-Shulman's case with $a = a^*$ playing the role of relaxation parameter.

If we put $a = a^*$ then the characteristic equation (19) and the frequency equation (20) coincide with the case corresponding to Lord-Shulman theory[11].

References

1. H. Deresiewicz, *J. Acoust. Soc. Am.* **29** (1957) 204-209.
2. F. J. Lockett, *J. Mech. Phys. Solids.* **7** (1958), 71-75.
3. P. Chadwick and I. N. Senddon, *J. Mech. Phys. Solids* **6** (1958), 223-30.
4. P. Chadwick, *Thermo-elasticity. The Dynamical Theory. Progress in Solid Mechnnics,* Vol. 1, Chapter 6. North-Holland Publishing Co. Amsterdam 1960.
5. S. K. Chakraborty and R. P. Pal, *Pure Appl. Geophy.* **76** (1969), 79-86.
6. M. A. Biot, *J. Appl. Phys.* **27** (1956), 240-53.
7. M. Lessen, *J. Mech. Phys. Solids* **5** (1956), 57-61.
7. H. W. Lord and Y. Shulnan, *J. Mech. Phys. Solids* **15** (1967), 299-309.
9. A. E. Green and K. A. Lindsay, *J. Elasticity* 2 (1972), 1-7.
10. A. E. Green, *Mathematica* **19** (1972), 69-75.
11. A. Nayfeh and S. Nemat-Nasser, *Acta Mech.* **12** (1971) 53-69.
12. V. K. Agarwal, *Acta Mech.* **31** (1979), 185-98.