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INCOMING WATER WAVES AGAINST A VERTICAL CLIFF IN A TWO-FLUID MEDIUM

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The two-dimensional problem of incoming waves against a vertical cliff in a two-fluid medium is considered in this paper. The velocity potentials describing the motion in each of the fluids are obtained. Known results for the one-fluid medium are recovered when the density of the upper fluid is made to vanish.

1. INTRODUCTION

Stoker[2,3] considered the problem of incoming surface water waves in a deep ocean bounded on one side by a vertical cliff. He solved the two-dimensional problem by employing a technique based on the complex variable theory and the three-dimensional problem by a reduction procedure. No reflection of waves by the cliff is assumed so that a source/sink type behaviour of the potential function there is necessary to account for this and as such the wave amplitude is required to be logarithmically infinite at the shore line. However, if the effect of surface tension at the free-surface is taken into account, this is not necesary, so that the potential remains regular there. Packham[1] solved the corresponding two-dimensional problem by using a technique based on the Fourier sine transform.

The present paper is concerned with the two-dimensional problem of incoming internal waves at the interface of a two-fluid medium against a vertical cliff in the presence of interfacial tension thereby generalising Packham's[1] problem. The velocity potentials in both the fluids are obtained in a straight forward manner.

2. STATEMENT AND SOLUTION OF THE PROBLEM

We consider the irrotational motion of two inviscid incompressible fluids of infinite horizontal extent towards the right and bounded by a vertical cliff on the left and of densities ρ_1 and ρ_2 ($< \rho_1$) respectively under the action of gravity only. The lower fluid extends infinitely downwards while the upper extends infinitely upwards. We choose a rectangular cartesian co-ordinate system such that the y-axis points vertically downwards into the lower fluid, then the cliff is the plane $x = 0$ and $y = 0$, $x \geq 0$ is the position of the undisturbed interface. The velocity potentials $\Phi_1(x, y, t)$ and $\Phi_2(x, y, t)$ for the lower and upper fluids respectively, under assumption of time-harmonic motion, can be described by

$$\Phi_j(x, y, t) = \text{Re}\,[\varphi_j(x, y \exp)(-i\sigma t)]\ (j = 1, 2)$$

where σ is the circular frequency. Then φ_j' s satisfy

$$\nabla^2 \varphi_j = 0 \text{ in the respective region} \qquad \ldots(2.1)$$

$$\left.\begin{aligned} &\varphi_{1y} = \varphi_{2y} \\ &K\varphi_1 + \varphi_{1y} - s(K\varphi_2 + \varphi_{2y}) + M \begin{Bmatrix} \varphi_{1yyy} \\ \varphi_{2yyy} \end{Bmatrix} = 0 \text{ on } y = 0,\ x > 0 \end{aligned}\right\} \qquad \ldots(2.2)$$

$$\varphi_{jx} = 0 \text{ on } x = 0\ (y > 0 \text{ for } j = 1 \text{ and } y < 0 \text{ for } j = 2) \qquad \ldots(2.3)$$

φ_j's remain finite at the origin .. (2.4)

$$\nabla \varphi_j \to 0 \text{ as } y \to \pm\infty \qquad \ldots(2.5)$$

(the upper sign is for $j = 1$ and the lower sign for $j = 2$) and finally

$$\varphi_j \sim \pm \exp(\mp k_0 y - ik_0 x) \text{ as } x \to \infty \qquad \ldots(2.6)$$

where k_0 is the unique positive real root of

$$k(1 + M' k^2) - L = 0 \qquad \ldots(2.7)$$

$$M' = M/(1 - s),\ L = K(1 + s)/(1 - s),\ K = \sigma^2/g$$

$$s = \rho_2/\rho_1,\ M = T/(\rho_1 g).$$

T being the interfacial tension and g the gravity.

Here, (2.1) is the equation of continuity in either of the fluids, (2.2) is the linearized kinematical and dynamical conditions at the interface, (2.3) is the condition at the cliff, (2.4) is the condition at the shore-line, (2.5) is the condition of no motion at infinite depth and height, and (2.6) is due to the incoming nature of the waves as $x \to \infty$ moving towards the cliff.

To solve for $\varphi_j(x, y)$ we write

$$\varphi_j(x, y) = \pm 2 \exp(\mp k_0 y) \cos(k_0 x) + \psi_j(x, y). \qquad \ldots(2.8)$$

Then ψ_j's satisfy the same eqns (2.1) to (2.5) as satisfied by φ_j's and

$$\psi_j(x, y) \to \mp \exp(\mp k_0 y + i k_0 x) \text{ as } x \to \infty. \qquad \ldots(2.9)$$

(2.9) states that ψ_j's behave as outgoing waves as $x \to \infty$.

Solution of ψ_j's satisfying (2.1) to (2.5) is given by

$$\psi_j(x,y) = \pm c \int_0^\infty \frac{\exp(\mp k_y)}{k(1+M'k^2)-L} \cos(kx)\, dk\ (j = 1,2) \qquad (2.10)$$

where the path of integration is indented below the pole at $k = k_0$ to account for the outgoing nature of ψ_j's as $x \to \infty$ and c is a constant to be chosen such that (2.9) is satisfied. We may note that ψ_j's given by (2.10) remain finite as $(x^2 + y^2)^{1/2} \to 0$ so long as $T > 0$. However, for $T = 0$, (2.10) exhibit a logarithmic singularity as $r \to 0$ (cf. Yu and Ursell[4]) which accounts for a source/sink type behaviour at the shore-line in the absence of interfacial tension as stated in the introduction.

Alternative representation for $\psi_j\,(x, y)$'s are given by

$$\psi_j\,(x, y) = \pm\, c\left[\frac{\pi i}{1+3M\;k_0^2}\exp\,(\mp\, k_0\, y + ik_0\, x)\right.$$

$$\left. + \int_0^\infty \frac{\exp\,(-kx)}{k^2\,(1-M'k^2)^2 + L^2}\{k\,(1-M'k^2)\cos ky \mp L\sin ky\}\,dk\right]. \quad \text{...(2.11)}$$

To satisfy (2.9), we must choose

$$c = \frac{i}{\pi}(1 + 3M'\,k_0^2). \quad \text{...(2.12)}$$

Thus we obtain finally,

$$\Phi_j\,(x, y, t) = \pm\,\exp\,(\mp\, k_0\, y)\cos\,(k_0\, x + \sigma t)$$

$$\pm\,\frac{\sin\sigma t}{\pi}\,(1 + 3M'\,k_0^2)\int_0^\infty$$

$$\frac{\exp\,(-kx)\,\{k\,(1-M'\,k^2)\cos ky \mp L\,\sin ky\}}{k^2\,(1-M'\,k^2)^2 + L^2}\,dk. \quad \text{...(2.13)}$$

In the absence of interfacial tension (2.13) reduces to

$$\Phi_j\,(x, y, t) = \pm\,\exp\,(\mp\, Ly)\cos\,(Lx + \sigma\, t)$$

$$\pm\,\frac{\sin\,\sigma t}{\pi}\int_0^\infty \frac{\exp\,(-kx)}{k^2 + L^2}\,\{k\cos ky \mp L\sin ky\}\,dk. \quad \text{...(2.14)}$$

Putting $s = 0$ in the expression for Φ_1 in (2.14), the potential function obtained by Stoker[2,3] is recovered. Again, putting $s = 0$ in Φ_1 into (2.13) we obtain

$$\Phi_1\,(x, y, t) = \exp\,(-k_0\, y)\cos\,(k_0\, x + \sigma t)$$

$$+\,\frac{\sin\,\sigma t}{\pi}\,(1 + 3Mk_0^2)\int_0^\infty \frac{\exp\,(-\,kx)}{k^2\,(1 - Mk^2)^2 + K^2}$$

$$\{k\,(1 - Mk^2)\cos ky - K\sin ky\}\,dk$$

which was derived by Packham[1] using a different approach.

4. Discussion

Potential functions representing incoming progressive waves against a vertical cliff in a two-fluid medium are obtained. Putting $s = 0$, the result for a deep ocean is recovered, more specifically, Φ_1 reduces to Stoker's[2,3] result when the surface tension is ignored and to Packham's[1] result when this is included. Also the potential functions in a two-fluid medium in the absence of interfacial tension are derived by simply putting $T = 0$. This problem can be further generalised to include the cases where the lower fluid is of uniform finite depth and/or the upper fluid is bounded by a horizontal rigid lid or a free surface.

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