Indian J. pure appl. Math., **20** (4) : 307-329, April 1989

ON THE UNIFORM STABILITY OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH COMPLEX COEFFICIENTS

Z. ZAHREDDINE

Department of Mathematics, U.A.E. University, Al-Ain, P. O. Box 15551
United Arab Emirates

(*Received* 9 *June* 1988)

In a recent paper[4], necessary and sufficient conditions for the asymptotic stability of a system of differential equations of dimension at most 4 with complex coefficients were established. In the present work, other types of stability are considered, the results of Zahreddine and Elshehawey[4] are extended, and necessary and sufficient condition for the stability and uniform stability of the above systems are established.

1. INTRODUCTION

Consider the homogeneous, first order linear system of ordinary differential equations of n-dimensions $X' = AX$ where A is an $n \times n$ complex constant matrix and $X(t)$ is a column vector of the n dependent variables. The characteristic equation is a polynomial equation of degree n whose roots are the eigenvalues of A.

We follow the definitions of asymptotic stability, uniform stability and stability as given in Jordan and Smith[2], and we note that when A is constant, then stability implies uniform stability [Jordan and Smith[2], remarks following definition (9.3)].

By (Jordan and Smith[2], Theorem 9.3), the question of stability of the system $X' = AX$ is related to the nature of the eigenvalues of A, when A is complex, more details may be found in Boyce and Diprima[1]. With the help of Hurwitz polynomials and positive functions, we were able to establish (Section 3 of Zahreddine and Elshehawey[4]) necessary and sufficient conditions for the asymptotic stability of the system $X' = AX$ where A is a complex matrix of dimension at most 4. We intend to include not only asymptotic stability, but also stability and therefore uniform stability. We extend the results of Zahreddine and Elshehawey[4] and establish necessary and sufficient conditions for the stability and uniform stability of the systems described above.

Before we proceed, we recall some basic definitions and facts.

2. DEFINITIONS AND NOTATIONS

Definition 2.1—The polynomial $f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n$ with complex coefficients is a Hurwitz polynomial if all its roots have negative real parts.

Definition 2.2—If $g(\lambda)$ is a rational function, its paraconjugate is defined by $g^*(\lambda) = \overline{g(-\bar{\lambda})}$, where $\bar{\lambda}$ denotes the complex conjugate of λ.

When $f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots\ldots + a_{n-2} \lambda^2 + a_{n-1} \lambda + a_n$

then

$$f^*(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \bar{a}_1 \lambda^{n-1} + \ldots + \bar{a}_{n-2} \lambda^2 - \bar{a}_{n-1} \lambda + \bar{a}_n.$$

Definition 2.3—A function $h(\lambda)$ is said to be positive if $\operatorname{Re} h(\lambda) > 0$ whenever $\operatorname{Re} \lambda > 0$.

The study of Hurwitz polynomials may be reduced to that of positive functions Levinson and Redheffer[3], (Theorem 5.1), and, by Theorem 5.2 of Levinson and Redheffer[3] any rational function h such that h and $-h^*$ are both positive can be written in the form :

$$h(\lambda) = a + b\lambda + \frac{b_1}{\lambda - iw_1} + \frac{b_2}{\lambda - iw_2} + \ldots + \frac{b_n}{\lambda - iw_n} \qquad \ldots(1)$$

where $\operatorname{Re} a = 0$, $b \geqslant 0$, $b_k \geqslant 0$ and where the W_j are distinct real numbers. This form will be referred to very frequently in the arguments that follow. According to the proof of Theorem 5.2 in Levinson and Redheffer[3], the W_j are the roots of $1/h$ and since they are distinct, it is easy to show that the expansion (1) of $h(\lambda)$ is unique.

3. Stability of the System $X' = AX$

We need the following two lemmas.

Lemma 3.1—If f and f^* have only one root in common, then it must have zero real part.

Proof : Write $f(\lambda)$ and $f^*(\lambda)$ in the factored forms :

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots\ldots (\lambda - \lambda_n)$$

and

$$f^*(\lambda) = (-1)^n (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2) \ldots\ldots (\lambda + \bar{\lambda}_n).$$

Suppose that λ_1 is the only common root to f and f^*.

If $\lambda_1 = -\bar{\lambda}_j$ for some j between 2 and n, then $\lambda_j = -\bar{\lambda}_1$. Since $-\bar{\lambda}_1$ is a root to f^*, λ_j becomes another common root to f and f^* leading to a contradiction. Hence $\lambda_1 = -\bar{\lambda}_1$, or $\operatorname{Re} \lambda_1 = 0$.

Lemma 3.2—λ_1 is a common root to f and f^* if and only if it is a common root to $f + f^*$ and $f - f^*$.

Proof : It follows from the observation that, $f = \frac{1}{2}[(f + f^*) + (f - f^*)]$ and $f^* = \frac{1}{2}[(f + f^*) - (f - f^*)]$.

Now, we consider the case where A is a 2×2 complex matrix with characteristic polynomial

$$f(\lambda) = \lambda^2 + a_1 \lambda + a_2. \quad \text{Let } f^*(\lambda) = \lambda^2 - \bar{a}_1 \lambda + \bar{a}_2$$

be the paraconjugate of f.

Theorem 3.1—The system $X' = AX$ where X is a 2×2 complex matrix with no repeated zero eigenvalue, is stable if and only if one of the following two conditions hold :

1. $\operatorname{Re} a_1 > 0$ and $\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) - (\operatorname{Im} a_2)^2 \geqslant 0$

2. $\operatorname{Re} a_1 = \operatorname{Im} a_2 = 0$ and $a_1^2 - 4a_2 \leqslant 0$, where asymptotic stability occurs only when both inequalities in 1, are strict.

PROOF : The case of asymptotic stability is settled by Theorem 3.1 Zahreddine and Elshehawey[4].

Uniform stability, or stability which is not asymptotic occurs only in each of the following cases :

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)$ where $\operatorname{Re} \lambda_2 < 0$

Case 2—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)$ where λ_2 is a non-zero real number. In both cases λ_1 is a possibly zero real number[2] (Theorem 9.3).

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)$.

Define

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = \lambda - \lambda_2.$$

Since $\operatorname{Re} \lambda_2 < 0$ $g(\lambda)$ is a Hurwitz polynomial,

$$g^*(\lambda) = -\frac{f^*(\lambda)}{\lambda - i\lambda_1} = -(\lambda + \bar{\lambda}_2).$$

Let

$$h(\lambda) = -\frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$$

g and g^* have no roots in common[4] (Lemma 3.1)

Therefore h is a positive function[3] (Theorem 5.1).

Since $h^*(\lambda) = -h(\lambda)$, $-h^*$ is positive, and $h(\lambda)$ can be expanded uniquely as in (1), (Levinson and Redheffer[3], Theorem 5.2)

$$h(\lambda) = \frac{\lambda^2 + \operatorname{Im} a_1 \lambda + \operatorname{Re} a_2,}{\operatorname{Re} a_1 \lambda\, i \operatorname{Im} a_2} = \frac{2\lambda + \lambda_2 - \lambda_2}{-(\lambda_2 + \bar{\lambda}_2)} \qquad \text{...(2)}$$

If Re $a_1 = 0$, then $h(\lambda)$ becomes a seeond degree polynomial in λ which obviously is not true. Hence Re $a_1 \neq 0$. We execute a long division to bring $h(\lambda)$ to the form

$$h(\lambda) = \frac{i}{(\text{Re } a_1)^2} (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{\text{Re } a_1} \lambda$$

$$+ \frac{\text{Re } a_2 + \dfrac{\text{Im } a_2}{(\text{Re } a_1)^2} . (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2)}{\text{Re } a_1 \lambda + i \text{ Im } a_2}$$

which when compared to (1) leads to Re $a_1 > 0$.

Since $h(\lambda)$ is a first degree polynomial in λ, the remainder of this division must be zero, implying that Re a_1 $Re(a_1 \bar{a}_2) - (\text{Im } a_2)^2 = 0$.

Conversely, assume that Re $a_1 > 0$ and Re a_1 Re $(a_1 \bar{a}_2) - (\text{Im } a_2)^2 = 0$. Take $h(\lambda)$ as in the first of the proof :

$$h(\lambda) = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)} = \frac{\lambda^2 + i \text{ Im } a_1 \lambda + \text{Re } a_2}{\text{Re } a_1 \lambda + i \text{ Im } a_2}$$

$$= \frac{i}{(\text{Re } a_1)^2} (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{(\text{Re } a_1)} \lambda \quad .. (3)$$

for the remainder of this division is zero.

This shows that $f + f^*$ and $f - f^*$ have a common root, thus implying that f and f^* have a common root (Lemma 3.2) which must be unique, for otherwise $f = f^*$ leading to Re $a_1 = 0$. Therefore the common root to f and f^* has zero real part (Lemma 3.1) call it $i\lambda_1$, where λ_1 real.

Let

$$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)$$

since Re $a_1 > 0$, by (3), $h(\lambda)$ is positive function. (Theorem 5.2).

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = \lambda - \lambda_2$$

then

$$g^*(\lambda) = - \frac{f^*(\lambda)}{\lambda - i\lambda_1} = -(\lambda + \bar{\lambda}_2)$$

and

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} .$$

Now g and g^* have no common roots, and h positive. Therefore g is a Hurwitz polynomial (Levinson and Redheffer[3], Theorem 5.1), leading to Re $\lambda_2 < 0$

Case 2— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)$.

$$f(\lambda) = \lambda^2 - i(\lambda_1 + \lambda_2)\lambda - \lambda_1 \lambda_2 = \lambda^2 + a_1 \lambda + a_2$$

so $\lambda_1 + \lambda_2 = i\, a_1$ and $\lambda_1 \lambda_2 = - a_2$.

Hence Re a_1 = Im $a_2 = 0$. it is easy to check the identity $(\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2 = (\lambda_1 - \lambda_2)^2 \geqslant 0$.

Therefore $4a_2 - a_1^2 \geqslant 0$

conversely, assume Re a_1 = Im $a_2 = 0$, and $4a_2 - a_1^2 \geqslant 0$.

Consider the quadratic equation with real coefficients $X^2 - ia_1 X - a_2 = 0$. Its discriminant $- a_1^2 + 4a_2 \geqslant 0$, so it has two real roots λ_1 and λ_2 such that :

$\lambda_1 + \lambda_2 = ia_1$ and $\lambda_1 \lambda_2 = - a_2$. It is easy to verify that $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)$.

Consider now, the system $X = AX$ where A is a 3×3 complex matrix with characteristic polynomial

$f(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$, and let $f^*(\lambda) = - \lambda^3 + \bar{a}_1 \lambda^2 - \bar{a}_2 \lambda + \bar{a}_3$ be the paraconjugate of f.

Define the numbers a, b and c by :

$$a = \text{Re}\, a_1 \,\text{Re Re}\,(a_1 \bar{a}_2 - a_3) - (\text{Im}\, a_2)^2,$$

$$b = \text{Re}\, a_1 \,\text{Re}\,(a_2 \bar{a}_3) - (\text{Re}\, a_3)^2, \text{ and}$$

$$c = \text{Re}\, a_1 \,\text{Im}\,(\bar{a}_1 a_3) + a_3 \,\text{Im}\, a_2.$$

Theorem 3.2—The system $X' = AX$ where A is a 3×3 complex matrix no repeated zero eigenvalue, is stable if and only if one of the following three conditions hold :

1. Re $a_1 > 0$, $a > 0$ and $ab - c^2 \geqslant 0$
2. Re $a_1 > 0$, $a = c = 0$ and 4 Re a_1 Re $(a_1 \bar{a}_2) - 3\,(\text{Im}\, a_2)^2 > 0$
3. Re a_1 = Im a_2 = Re $a_3 = 0$ and there exists a number α with Re $\alpha = 0$, $f(\alpha) = 0$ and satisfying $3\alpha^2$ and $2 a_1 \alpha - a_1^2 + 4a_2 \geqslant 0$, where asymptotic stability occurs only when all the inequalities in 1, are strict.

PROOF : The case of asymptotic stability is settled by (Zahreddine and Elshehawey[2], Theorem 3.2). Uniform stability, or stability which is not asymptotic, occurs only in each of the following cases:

Case 1— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$ where Re $\lambda_2 < 0$, Re $\lambda_3 < 0$.

Case 2— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$ where λ_2 is a non zero real number and Re $\lambda_3 < 0$.

Case 3— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$ where λ_2 and λ_3 are non-zero real numbers.

In all three cases, λ_1 denotes a possibly zero real number (Theorem 9.3 of Jordan and Smith[2]).

Case 1— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$

Define

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = (\lambda - \lambda_2)(\lambda - \lambda_3)$$

g is a Hurwitz polynomial for Re $\lambda_2 < 0$ and Re $\lambda_3 < 0$

$$g^*(\lambda) = - \frac{f^*(\lambda)}{\lambda - i\lambda_1} = (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)$$

let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$$

g and g^* have no roots in common (Lemma 3.1 of Zahreddine and Elshehawey[4]) and therefore h is a positive function, (Theorem 5.1 of Levinson and Redheffer[3]).

Easy to see that

$$h(\lambda) = \frac{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}{\lambda^3 + i \text{ Im } a_1 \lambda^2 + \text{Re } a_2 \lambda + i \text{ Im } a_3} .$$

For any complex number $\gamma \neq 0$ it is easily checked that Re γ and Re $(1/\gamma)$ have the same sign. Hence $1/h$ is positive if h is positive. And since $h^*(\lambda) = - h(\lambda)$, then $(1/h)^* = - 1/h$, hence $- (1/h)^*$ is positive. Therefore the function

$$\frac{1}{h(\lambda)} = \frac{\lambda^3 + i \text{ Im } a_1 \lambda^2 + \text{Re } a_2 \lambda + i \text{ Im } a_3}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3} \qquad \ldots(4)$$

can be written as in (1), (Theorem 5.2 of Levinson and Redheffer[3])

Also,

$$\frac{1}{h(\lambda)} = \frac{\lambda^2 - i \text{ Im } (\lambda_2 + \lambda_3)\lambda + \text{Re } (\lambda_2 \lambda_3)}{- \text{Re } (\lambda_2 + \lambda_3)\lambda + i \text{ Im } (\lambda_2 \lambda_3)} \qquad \ldots(5)$$

where Re $(\lambda_2 + \lambda_3) \neq 0$.

By comparing (4) and (5) we conclude that Re $a_1 \neq 0$. In (4) we execute a long division :

$$\frac{1}{h(\lambda)} = \frac{i}{(\text{Re} a_1)^2} (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{\text{Re } a_1} \lambda$$

$$+ \frac{R}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3} ,$$

where

$$R = \frac{1}{\text{Re } a_1} (a \lambda + ic)$$

a and c are defined above.

Re $a_1 > 0$ (Theorem 5.2 of Levinson and Redheffer[3]). Also Theorem 5.2 of Levinson and Redheffer[3] and the remarks that follow as illustrated by Example 5.1 in Levinson and Redheffer[3] imply that $K(\lambda) = \dfrac{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}{R}$ is a positive function, and so is $- K^*(\lambda)$. Therefore $K(\lambda)$ has form (1), where again it can be shown that $a/(\text{Re } a_1)^2$ the coefficient of λ in R is non-zero.

By (5) $1/h$ reduces to a second degree polynomial in λ over a first degree one, which in turn implies that

$\dfrac{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}{R}$ appears as a first degree polynomial in λ and (2) of case 1, Theorem 3.1 has re-emerged but now with different coefficients. A repetition of that argument leads to $a > 0$ and $ab - c^2 = 0$.

Conversely, assume that Re $a_1 > 0$, $a > 0$ and $ab - c^2 = 0$

$$\frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)} = \frac{\lambda^3 + i \text{ Im } a_1 \lambda^2 + \text{Re } a_2 \lambda + i \text{ Im } a_3}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3} \quad \ldots(6)$$

$$= \frac{i}{(\text{Re } a_1)^2} (\text{Re } a_1 \text{ Im } a_1 - \text{Im } a_2) + \frac{1}{\text{Re } a_1} \lambda$$

$$+ \frac{R}{\text{Re } a_1 \lambda^2 + i \text{Im } a_2 \lambda + \text{Re } a_3} \lambda$$

where R as defined before.

$$ab - c^2 = 0 \text{ implies that } \frac{R}{\text{Re } a_1 \lambda^2 + i \text{ Im } a_2 \lambda + \text{Re } a_3}$$

reduces to a non-zero constant over a first degree polynomial in λ. Therefore $f + f^*$ and $f - f^*$ have a common root which must be unique, for otherwise $(f - f^*)/(f + f^*)$ reduces to a first degree polynomial in λ, leading to $R = 0$ which is certainly not true since $a > 0$. Therefore f and f^* have only one root in common (Lemma 3.2), which must have zero real part (Lemma 3.1), call it $i \lambda_1$.

Let

$$f(\lambda) = (\lambda - i \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

Now if

$$h(\lambda) = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}, \text{ then } \frac{1}{h(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$$

is a positive function since Re $a_1 > 0$ and $a > 0$. So h is also a positive function.

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_1} = (\lambda - \lambda_2)(\lambda - \lambda_3)$$

then

$$g^*(\lambda) = -\frac{f^*(\lambda)}{\lambda - i\lambda_1} = (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)$$

and

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$$

g and g^* do not vanish simultaneously since $i\lambda_1$ is the only common root of f and f^*, and because $h(\lambda)$ is positive we conclude that $g(\lambda)$ is a Hurwitz polynomial (Theorem 5.1, Levinson and Redheffer[3]) therefore Re $\lambda_2 < 0$ and Re $\lambda_3 < 0$

Case 2— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)} = \lambda - \lambda_3$$

is a Hurwitz polynomial since Re $\lambda < 0$

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)} = -(\lambda + \bar{\lambda}_3).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$$

h is a positive function (Lemma 3.1 Zahreddine and Elshehawey[4] and Theorem 5.1 of Levinson and Redheffer[3]), and so is $-h^* = h$ therefore $h(\lambda)$ takes from 1), (Theorem 5.2 of Levinson and Redheffer[3])

Also h can be written :

$$h(\lambda) = \frac{\lambda^3 + i\,\mathrm{Im}\,a_1\lambda^2 + \mathrm{Re}\,a_2\lambda + i\,\mathrm{Im}\,a_3}{\mathrm{Re}\,a_1\lambda^2 + i\,\mathrm{Im}\,a_2\lambda + \mathrm{Re}\,a_3} = \frac{2\lambda - (\lambda_3 - \bar{\lambda}_3)}{-(\lambda_3 + \bar{\lambda}_3)}. \quad \ldots(7)$$

It is easy to see that Re $a_1 \neq 0$. The first part of (7) leads to

$$h(\lambda) = \frac{i}{(\mathrm{Re}\,a_1)^2}(\mathrm{Re}\,a_1\,\mathrm{Im}\,a_1 - \mathrm{Im}\,a_2) + \frac{1}{\mathrm{Re}\,a_1}\lambda + \frac{R}{\mathrm{Re}\,a_1\lambda + i\,\mathrm{Im}\,a_2\,\lambda + \mathrm{Re}\,a_3}.$$

By (7), $R = 0$. Since $R = \frac{1}{(\mathrm{Re}\,a_1)^2}(a\lambda + ic)$, then $a = c = 0$.

$\operatorname{Re} a_1 > 0$ for h is positive (Theorem 5.2 of Levinson and Redheffer[3]).

It remains to verify the inequality $4 \operatorname{Re} a_1 \operatorname{Re} (a_1 \bar{a}_2) - 3 (\operatorname{Im} a_2)^2 \geqslant 0$.

Since

$$f(\lambda) = (\lambda - i \lambda_1)(\lambda - i \lambda_2)(\lambda - \lambda_3) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

then

$$i \lambda_1 + i \lambda_2 + \lambda_3 = - a_1,\ i \lambda_1 \lambda_3 + i \lambda_3 \lambda_2 - \lambda_1 \lambda_2 = a_2,\ \lambda_1 \lambda_2 \lambda_3 = a_3.$$

The real part of the first relation, imaginary parts of the second and third lead respectively to :

$$\operatorname{Re} \lambda_3 = - \operatorname{Re} a_1,\ (\lambda_1 + \lambda_2) \operatorname{Re} \lambda_3 = \operatorname{Im} a_2 \text{ and } \lambda_1 \lambda_2 \operatorname{Im} \lambda_3 = \operatorname{Im} a_3.$$

Therefore

$$\lambda_1 + \lambda_2 = - \frac{\operatorname{Im} a_2}{\operatorname{Re} a_1} \text{ and } \lambda_1 \lambda_2 = - \frac{\operatorname{Re} a_3}{\operatorname{Re} a_1} .$$

So λ_1 and λ_2 are the roots of the quadratic equation

$$\operatorname{Re} a_1 X^2 + \operatorname{Im} a_2 X - \operatorname{Re} a_3 = 0.$$

Since λ_1 and λ_2 are real $(\operatorname{Im} a_2)^2 + 4 \operatorname{Re} a_1 \operatorname{Re} a_3 \geqslant 0$, but $a = \operatorname{Re} a_1 \operatorname{Re} (a_1 \bar{a}_2 - a_3) - (\operatorname{Im} a_2)^2 = 0$ which implies $(\operatorname{Im} a_2)^2 + 4 \operatorname{Re} a_1 \operatorname{Re} a_3 = 4 \operatorname{Re} a_1 \operatorname{Re} (a_1 \bar{a}_2) - 3 (\operatorname{Im} a_2)^2$, leading therefore to the conclusion.

Conversely, suppose that $\operatorname{Re} a_1 \geqslant 0$, $a = c = 0$ and $4 \operatorname{Re} a_1 \operatorname{Re} (a_1 \bar{a}_2) - 3 (\operatorname{Im} a_2)^2 \geqslant 0$

$$\frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)} = \frac{\lambda^3 + i \operatorname{Im} a_1 \lambda^2 + \operatorname{Re} a_2 \lambda + i \operatorname{Im} a_3}{\operatorname{Re} a_1 \lambda^2 + i \operatorname{Im} a_2 \lambda + \operatorname{Re} a_3} \quad \ldots(8)$$

$$= \frac{1}{(\operatorname{Re} a_1)^2} (\operatorname{Re} a_1 - \operatorname{Im} a_1 - \operatorname{Im} a_2) + \frac{1}{\operatorname{Re} a_1} \lambda$$

for the remainder of this division is zero, since $a = c = 0$. Therefore $f - f^*$ and $f + f^*$ have two common roots, implying that f and f^* also have two common roots (Lemma 3.2), call them λ_1 and λ_2.

Let

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

λ_3 cannot be another common root to f and f^*, for then $f = - f^*$ leading to $\operatorname{Re} a_1 = 0$

write

$$f^*(\lambda) = - (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)$$

if $\lambda_1 = -\bar{\lambda}_3$ then $\lambda_3 = -\bar{\lambda}_1$, ands ince $-\bar{\lambda}_1$ is a root of f^*, λ_3 becomes a common root to f and f^* which is impossible. The same applies on λ_2.

Therefore we have only two possibilities either $\lambda_1 = -\bar{\lambda}_1$, leading to $\lambda_2 = -\bar{\lambda}_2$

or $\lambda_1 = -\bar{\lambda}_2$ which is equivalent to $\lambda_2 = -\bar{\lambda}_1$.

In both cases, we get $f^*(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda + \bar{\lambda}_3)$.

Let

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = \lambda - \lambda_3$$

so

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda+\bar{\lambda}_1)(\lambda - \bar{\lambda}_2)} = \frac{f^*(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = -(\lambda + \bar{\lambda}_3).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}.$$

In (8) above, $\mathrm{Re}\, a_1 > 0$ and $h^* = -h$ therefore $h(\lambda)$ is a positive function (Theorem 5.2 of Levinson and Redheffer[3]), and since g and g^* have no roots in common, g is a Hurwitz polynomial, (Theorem 5.1 of Levinson and Redheffer[3]). Hence $\mathrm{Re}\, \lambda_3 < 0$.

It remains to determine the nature of λ_1 and λ_2. When $\lambda_1 = -\bar{\lambda}_1$ and $\lambda_2 = -\bar{\lambda}_2$, then $\mathrm{Re}\, \lambda_1 = \mathrm{Re}\, \lambda_2 = 0$.

In this case the inequality $4\ \mathrm{Re}\, a_1\, \mathrm{Re}\,(a_1 \bar{a}_2) - 3\,(\mathrm{Im}\, a_2)^2 \geqslant 0$, is easily verified as in the previous section.

Consider now the second possibility where $\lambda_1 = -\bar{\lambda}_2$,

or $\lambda_2 = -\bar{\lambda}_1$. This leads to $\mathrm{Re}\, \lambda_1 = -\mathrm{Re}\, \lambda_2$ and

$\mathrm{Im}\, \lambda_1 = \mathrm{Im}\, \lambda_2$. Call $\beta = \mathrm{Re}\, \lambda_1$ and $\gamma = \mathrm{Im}\, \lambda_1$, so

$$\lambda_1 = \beta + i\gamma, \text{ and } \lambda_2 = -\beta + i\gamma.$$

$$f(\lambda) = [\lambda - (\beta + i\gamma)]\,[\lambda - (-\beta + i\gamma)].\,(\lambda - \lambda_3)$$

$$= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \text{ Then :}$$

$$2i\gamma + \lambda_3 = -a_1,\ 2i\gamma\,\lambda_2 - \beta^2 - \gamma^2 = a_2,\ (\beta^2 + \gamma^2)\,\lambda_3 = a_3.$$

Obtain λ_3 from the first relation and substitute it in the second to get : $3\gamma^2 - 2ia_1\gamma - \beta^2 - a_2 = 0$ whose real and imaginary parts produce :

$2\gamma \operatorname{Re} a_1 + \operatorname{Im} a_2 = 0$ and $\beta^2 = 3\gamma^2 + 2 \operatorname{Im} a_1 \gamma - \operatorname{Re} a_2$.

In the last relation, we substitute $\gamma = -\operatorname{Im} a_2/2\operatorname{Re} a_1$, to finally have

$$\beta^2 = \frac{3(\operatorname{Im} a_2)^2 - 4\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2)}{4(\operatorname{Re} a_1)^2}.$$

But this requires, since $\beta^2 \geqslant 0$, that $3(\operatorname{Im} a_2)^2 - 4\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) \geqslant 0$, and by assumption, we have $3(\operatorname{Im} a_2)^2 - 4\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_2) \leqslant 0$, when we compromise these two inequalities, we end up with $\beta = 0$ leading to $\operatorname{Re} \lambda_1 = 0$ and $\operatorname{Re} \lambda_2 = 0$.

Case 3—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$

$= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$.

Therefore

$$\lambda_1 + \lambda_2 + \lambda_3 = ia_1,\ \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = -a_2 \text{ and } \lambda_1\lambda_2\lambda_3 = -i a_3.$$

These relations imply $\operatorname{Re} a_1 = \operatorname{Im} a_2 = \operatorname{Re} a_3 = 0$.

The first and second relations lead to $\lambda_2 + \lambda_3 = ia_1 - \lambda_1$ and $\lambda_2\lambda_3 = -a_2 - \lambda_1(\lambda_2 + \lambda_3) = -a_2 + \lambda_1^2 - ia_1\lambda_1$.

So λ_2 and λ_3 are the roots of the quadratic equation with real coefficients $X^2 + (\lambda_1 - ia_1)X + (\lambda_1^2 - ia_1\lambda_1 - a_2) = 0$.

Since λ_2 and λ_3 are real, $(\lambda_1 - ia_1)^2 - 4(\lambda_1^2 - ia_1\lambda_1 - a_2) > 0$,

or $-3\lambda_1^2 + 2Ia_1\lambda_1 - a_1^2 + 4a_2 \geqslant 0$. Letting $\alpha = \lambda_1$, we get

$3\alpha^2 + 2a_1\alpha - a_1^2 + 4a_2 \geqslant 0$.

Conversely, suppose $\operatorname{Re} a_1 = \operatorname{Im} a_2 = \operatorname{Re} a_3 = 0$, and there exists α with $\operatorname{Re} \alpha = 0$, $f(\alpha) = 0$ and $3\alpha^2 + 2a_1\alpha - a_1^2 + 4a_2 \geqslant 0$, form the quadratic equation with real coefficients.

$X^2 - i(\alpha + a_1)X - (\alpha^2 + a_1\alpha + a_2) = 0$. The discriminant $3\alpha^2 + 2a_1\alpha - a_1^2 + ua_2 \geqslant 0$, therefore there exists two real numbers λ_2 and λ_3 such that :

$$\lambda_2 + \lambda_3 = i(\alpha + a_1) \text{ and } \lambda_2 . \lambda_3 = -(\alpha^2 + a_1\alpha + a_2)$$

also,

$$\lambda_2 . \lambda_3 = -a_2 - \alpha(\alpha + a_1) = -a_2 + i\alpha(\lambda_2 + \lambda_3).$$

Therefore,

$$-i\alpha + \lambda_2 + \lambda_3 = ia_1 \text{ and } -i\alpha\lambda_2 - i\alpha\lambda_3 + \lambda_2\lambda_3 = -a_2.$$

Since $f(\alpha) = 0$, then $\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 = 0$, and $\alpha(\alpha^2 + a_1\alpha + a_2) = -a_3$ which leads to $-\alpha\lambda_2\lambda_3 = -a_3$

or $\alpha\lambda_2\lambda_3 = a_3$. It is now straightforward to verify that

$f(\lambda) = (\lambda - \alpha)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$ and the proof is complete.

Finally, we consider the system $X' = AX$ where A is a 4×4 complex matrix with characteristic polynomial,

$$f(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

and let

$$f^*(\lambda) = \lambda^4 - \bar{a}_1\lambda^3 + \bar{a}_2\lambda^2 - \bar{a}_3\lambda + \bar{a}_4$$ be the paraconjugate of f.

Let a be as defined before Theorem 3.2, and define the numbers r, s and t as follows :

$$r = a.[\mathrm{Re}\, a_1 \mathrm{Re}(a_2\bar{a}_3 - a_1\bar{a}_4) - (\mathrm{Re}\, a_3)^2] - [\mathrm{Im}\, a_1 \mathrm{Re}(\bar{a_1} - a_3 a_4) + \mathrm{Re}\, a_3 \mathrm{Im}\, a_2]^2.$$

$$s = a.[\mathrm{Re}\, a_1 \mathrm{Re}(a_3\bar{a}_4) - (\mathrm{Im}\, a_4)^2] - [\mathrm{Re}\, a_1 \mathrm{Re}(a_1\bar{a}_4) - \mathrm{Im}\, a_2 \mathrm{Im}\, a_4]^2.$$

$$t = a.[\mathrm{Re}\, a_1 \mathrm{Im}(\bar{a}_2 a_4) - \mathrm{Re}\, a_3 \mathrm{Im}\, a_4] + [\mathrm{Re}\, a_1 \mathrm{Re}(a_1\bar{a}_4) - \mathrm{Im}\, a_2 \mathrm{Im} a_4].[\mathrm{Re}\, a_1 \mathrm{Im}(\bar{a}_1 a_3 - a_4) + \mathrm{Re}\, a_3 \mathrm{Im} a_2].$$

Theorem 3.3—The system $X' = AX$ where A is a 4×4 complex matrix with no repeated zero eigenvalue, is stable if and only if one of the following four conditions hold :

(1) $\mathrm{Re}\, a_1 > 0$, $a > 0$, $r > 0$ and $rs - t^2 \geq 0$.

(2) $\mathrm{Re}\, a_1 > 0$, $a > 0$, $r = t = 0$ and the inequality

$$4\,\mathrm{Re}\, a_1' \mathrm{Re}(a_1' \bar{a}_2') - 3(\mathrm{Im}\, a_2')^2 \geqslant 0 \text{ wher } a_1' = \gamma + a_1$$

$$a_2' = \gamma^2 + a_1\gamma + a_2, \text{ holds whenever } \mathrm{Re}\,\gamma < 0 \text{ and } f(\gamma) = 0.$$

(3) $\mathrm{Re}\, a_1 > 0$, $a = r = 0$, $\mathrm{Re}\, a_1 \mathrm{Re}(a_1\bar{a}_4) - \mathrm{Im}\, a_2 \,\mathrm{Im}\, a_4 = 0$, and there exists a real λ_0 with $f(i\lambda 0) = 0$ and satisfying

$$3\lambda_0^2 + 2\frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1}\lambda_0 - \frac{(\mathrm{Im}\, a_2)^2}{\mathrm{Re}\, a_1)^2} - \frac{4\,\mathrm{R}\, a_3}{\mathrm{Re}\, a_1} \leqslant 0$$

(4) Re a_1 = Im a_2 = Re a_3 = Im a_4 = 0 and there exist two numbers α and β with Re α = Re β = 0, $f(\alpha) = f(\beta) = 0$ and satisfying $3\alpha^2 + 2\alpha_1'\alpha - a_1' + 4a_2' \geqslant 0$ where

$$a_1' = \beta + a_1 \text{ and } a_2' = \beta^2 + a_1\beta + a_2$$

where asymptotic stability occurs only when all the inequalities in (1) are strict.

PROOF : These case of asymptotic stability is settled by Theorem 3.3 of Zahreddine and Elshehawey[4]. Non-asymptotic stability occurs only in each of the following cases.

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$ where Re $\lambda_i < 0$, $i = 2, 3, 4$.

Case 2—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$ where λ_2 is a non-zero real number, Re $\lambda_3 < 0$ and Re $\lambda_4 < 0$.

Case 3—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - \lambda_4)$, where λ_2 and λ_3 are non-zero real numbers, Re $\lambda_4 < 0$

Case 4—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - i\lambda_4)$, where λ_2, λ_3 and λ_4 are non-zero real numbers.

In all these cases, λ_1 denotes a possibly zero real number (Theorem 9.3, Jordan and Smith[2]).

Case 1—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\lambda - \lambda_4)$.

$$g^*(\lambda) = -\frac{f^*(\lambda)}{(\lambda - i\lambda_1)} = (\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) \text{ is a Hurwitz poly-}$$

nomial since Re $\lambda_i < 0$, $i = 2, 3, 4$.

$$g(\lambda) = -\frac{f(\lambda)}{\lambda - i\lambda_1} = -(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$$

g and g^* have no common roots (Lemma 3.1 of Zahreddine and Elshehawey[4]) and consequently h is a positive function (Theorem 5.1 of Levinson and Redheffer[3]).

$$h(\lambda) = \frac{\lambda^4 + i\,\text{Im}\,a_1\lambda^3 + \text{Re}\,a_2\lambda^2 + i\,\text{Im}\,a_3\lambda + \text{Re}\,a_4}{\text{Re}\,a_1\lambda^3 + i\,\text{Im}\,a_2\lambda^2 + \text{Re}\,a_3\lambda + i\,\text{Im}\,a_4}$$

Since h is positive, it takes form (1). Therefore Re $a_1 \neq 0$. By executing a long diiv-sion, we bring $h(\lambda)$ to the form :

$$h(\lambda) = \frac{1}{\text{Re}\, a_1}\lambda + \frac{i}{\text{Re}\, a_1}\left(\text{Im}\, a_1 - \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\right) + \frac{R'}{\text{Re}\, a_1 \lambda^3 + + i\, \text{Im}\, a_2 \lambda^2 + \text{Re}\, a_3 \lambda + i\, \text{Im}\, a_4}$$

where

$$R = \left[\left(\text{Re}\, a_2 - \frac{\text{Re}\, a_3}{\text{Re}\, a_1}\right) + \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\left(\text{Im}\, a_1 - \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\right)\right]\lambda^2 + i\left[\left(\text{Im}\, a_3 - \frac{\text{Im}\, a_4}{\text{Re}\, a_1}\right) - \frac{\text{Re}\, a_3}{\text{Re}\, a_1}\left(\text{Im}\, a_1 - \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\right)\right]\lambda + \text{Re}\, a_4 + \frac{\text{Im}\, a_4}{\text{Re}\, a_1}\left(\text{Im}\, a_1 - \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\right).$$

Since h is a positive function, $\text{Re}\, a_1 > 0$

Theorem 5.2 in Levinson and Redheffer[3] and the remarks that follow imply that

$$\frac{\text{Re}\, a_1 \lambda^3 + i\, \text{Im}\, a_2 \lambda^2 + \text{Re}\, a_3 \lambda + i\, \text{Im}\, a_4}{R'} \text{ is positive.}$$

Now,

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$$

$$= \frac{(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) + (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)}{(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) - (\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)}.$$

It is easy to see that $g + g^*$ is a second degree polynomial in λ, since the coefficient of λ^2 in $g + g^*$ is $-2\text{Re}\,(\lambda_2 + \lambda_3 + \lambda_4)$ which is non-zero.

Therefore the positive $\dfrac{\text{Re}\, a_1 \lambda^3 + i\, \text{Im}\, a_2 \lambda^2 + \text{Re}\, a_3 \lambda + i\, \text{Im}\, a_4}{R'}$ should also reduce to a second degree polynomial in λ over a degree one. So (4) and (5) of case 1., theorem 3.2 reappear but now with different and more complicated coefficients. The process here is rather lengthy, but the argument is entirely similar and when repeated leads to $a > 0$, $r > 0$ and $rs - t^2 = 0$.

Conversely, assume that $\text{Re}\, a_1 > 0$, $a > 0$, $r > 0$ and $rs - t^2 = 0$.

Take

$$\frac{f + f^*}{f - f^*} = \frac{\lambda^4 + i\, \text{Im}\, a_1 \lambda^3 + \text{Re}\, a_2 \lambda^2 + i\, \text{Im}\, a_3 \lambda + \text{Re}\, a_4}{\text{Re}\, a_1 \lambda^3 + i\, \text{Im}\, a_2 \lambda^2 + \text{Re}\, a_3 \lambda + i\, \text{Im}\, a_4}$$

$$= \frac{1}{\text{Re}\, a_1}\lambda + \frac{i}{\text{Re}\, a_1}\left(\text{Im}\, a_1 - \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\right) + \frac{R'}{\text{Re}\, a_1 \lambda^3 + i\, \text{Im}\, a_2 \lambda^2 + \text{Re} a_3 \lambda + i\, \text{Im}\, a_4}$$

where R' as defined in the first section. If we consider

$\frac{\text{Re}\, a_1\lambda^3 + i\,\text{Im}\, a_2\,\lambda^2 + \text{Re}\, a_3\,\lambda + i\,\text{Im}\, a_4}{R'}$ with the set of conditions $a > 0$, $r > 0$ and $rs - t^2 = 0$, we come to a position entirely similar to (6) in the converse of case 1, Theorem 3.2, with the same set of conditions but different and cumbersome coefficients. Another application of that argument leads to the fact that

$\text{Re}\, a_1\,\lambda^2 + i\,\text{Im}\, a_2\,\lambda^2 + \text{Re}\, a_3\,\lambda + i\,\text{Im}\, a_4$ and R' have only one root in common. The same applies on $f + f^*$ and $f - f^*$. By Lemma 3.2 f and f^* have only one common root which must have zero real part (Lemma 3.1) call it $i\,\lambda_1$.

Let $f(\lambda) = (\lambda - i\,\lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$.

If $h(\lambda) = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}$, then h is positive[3] (Theorem 5.2).

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\,\lambda_1}\ (\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4),$$

$$g^*(\lambda) = -\frac{f^*(\lambda)}{\lambda - i\,\lambda_1} = -(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)$$

so

$$h(\lambda) = \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)} = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$$

Since $i\,\lambda_1$ is the only common root to f and f^*, then g and g^* do not vanish simultaneously. Hence g is a Hurwitz polynomial[3] (Theorem 5.1). So $\text{Re}\,\lambda_i < 0$ for $i = 2, 3, 4$.

Case 2—$f(\lambda) = (\lambda - i\,\lambda_1)(\lambda - i\,\lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)} = (\lambda - \lambda_3)(\lambda - \lambda_4)$$

is a Hurwitz polynomial for $\text{Re}\,\lambda_i < 0$: $i = 3, 4$.

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda - i\,\lambda_1)(\lambda - i\,\lambda_2)} = (\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

Consider $h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}$

h is positive function [Lemma 3.1[4] and Theorem 5.1[3]] and so is

$$\frac{1}{h} = \frac{\lambda^4 + i\,\text{Im}\, a_1\,\lambda^3 + \text{Re}\, a_2\,\lambda^2 + i\,\text{Im}\, a_3\,\lambda + \text{Re}\, a_4}{\text{Re}\, a_1\,\lambda^3 + i\,\text{Im}\, a_2\,\lambda^2 + \text{Re}\, a_3\,\lambda + i\,\text{Im}\, a_4}$$

$1/h$ takes from (1), and that leads to $\text{Re}\, a_1 \neq 0$.

$$\frac{1}{h} = \frac{1}{\mathrm{Re}\, a_1}\lambda + \frac{i}{\mathrm{Re}\, a_1}(\mathrm{Im}\, a_1 - \frac{\mathrm{Im}\, a_2}{\mathrm{Re}\, a_1})$$

$$+ \frac{R'}{\mathrm{Re}\, a_1 \lambda^3 + i \,\mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re}\, a_3 \lambda + i \,\mathrm{Im}\, a_4}$$

where R' is already defined. By (Theorem 5.2 of Levinson and Redheffer[3]), Re $a_1 \rhd 0$. Also,

$$\frac{1}{h} = \frac{g(\lambda) + g^*(\lambda)}{g(\lambda) - g^*(\lambda)} = \frac{(\lambda - \lambda_3)(\lambda - \lambda_4) + (\lambda + \bar{\lambda}_3)(\lambda - \bar{\lambda}_4)}{(\lambda - \lambda_3)(\lambda - \lambda_4) - (\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)}.$$

The coefficient of λ in $g - g^*$ is $-2\,\mathrm{Re}\,(\lambda_3 + \lambda_4)$ which is non-zero. Hence $1/h$ reduces to a second degree polynomial in λ over a first degree one which implies that Re $a_1 \lambda^3 + i$ Im $a_2 \lambda^2 +$ Re $a_3 \lambda + i$ Im a_4 must be divisible by R'.

We are now in position (7) of case 2 of Theorem 3.2 but with different coefficients. The same argument leads to $r = t = 0$, and also to $a > 0$, since

$$\frac{\mathrm{Re}\, a_1 \lambda^3 + i \,\mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re}\, a_3 \lambda + i \,\mathrm{Im}\, a_4}{R'}$$

is a positive (Theorem 5.2 of Levinson and Redheffer[3]). Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$$

$$= \lambda^3 + a_1' \lambda^2 + a_2' \lambda + a_3'$$

where

$$a_1' = \lambda_4 + a_1, a_2' = \lambda_4^2 + a_1 \lambda_4 + a_2, a_3' = \lambda_4^3 + a_1 \lambda_4^2 + a_2 \lambda_4 + a_3$$

$$= -\frac{a_4}{\lambda_4}.$$

But $K(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - \lambda_3)$ is precisely case 2, Theorem 3.2, and therefore the inequality $4\,\mathrm{Re}\, a_1' \,\mathrm{Re}\,(a_1' \bar{a}_2') - 3\,(\mathrm{Im}\, a_2')^2 \geqslant 0$ is satisfied. The same can be proved with λ_3 replacing λ_4.

Conversely suppose Re $a_1 > 0$, $a > 0$, $r = t = 0$ and assume the above inequality holds with $a_1' = \gamma + a_1, a_2' = \gamma\lambda + a_1\gamma + a_2$, whenever, Re $\gamma < 0$ and $f(\gamma) = 0$.

Consider

$$\frac{f + f^*}{f - f^*} = \frac{\lambda^4 + i \,\mathrm{Im}\, a_1 \lambda^3 + \mathrm{Re}\, a_2 \lambda^2 + i \,\mathrm{Im}\, a_3 \lambda + \mathrm{Re}\, a_4}{\mathrm{Re}\, a_1 \lambda^2 + i \,\mathrm{Im}\, a_2 \lambda^2 + \mathrm{Re}\, a_3 \lambda + i \,\mathrm{Im}\, a_4}$$

(equation continued on p. 323)

$$= \frac{1}{\text{Re } a_1} \lambda + \frac{i}{\text{Re } a_1} (\text{Im } a_1 \frac{\text{Im } a_2}{\text{Re } a_1})$$

$$+ \frac{R'}{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4}. \quad \ldots(9)$$

Now

$$\frac{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4}{R'}$$ with $r = t = 0$ puts

us back in position (8) in the converse of case 2 of Theorem 3.2 with the same set of conditions. A similar argument implies that

$$\frac{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } _3 \lambda + i \text{ Im } a_4}{R'}$$ reduces to a first degree

polynomial in λ. Therefore $f + f^*$ and $f - f^*$ have two roots in common. By Lemma 3.2 f and f^* have two common roots, call them λ_1 and λ_2.

Let $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$. λ_1 and λ_2 are the only common roots to f and f^*, for if, for instance, λ_3 is another common root to f and f^*, so $f + f^*$ and $f - f^*$ would have three roots in common, which implies that $\frac{f + f^*}{f - f^*}$ becomes a first degree polynomial in which is not the case.

Consider $f^*(\lambda) = (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4)$.

If $\lambda_1 = -\bar{\lambda}_3$, then $\lambda_3 = -\bar{\lambda}_1$, and since $-\bar{\lambda}_1$ is a root of $f^*(\lambda)$, λ_3 becomes a common root to f and f^* which we proved to be impossible. Similarly λ_1 cannot equal $-\bar{\lambda}_4$. The same applies on λ_2. Therefore we have only two possibilities : either $\lambda_1 = -\bar{\lambda}_1$ leading to $\lambda_2 = -\bar{\lambda}_2$, or $\lambda_1 = -\bar{\lambda}_2$ which is equivalent to $\lambda_2 = -\bar{\lambda}_1$, In both cases, we get $f^*(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda + \lambda_3)(\lambda + \bar{\lambda}_4)$

$$\text{Let } g(\lambda) = \frac{f(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = (\lambda - \lambda_3)(\lambda - \lambda_4)$$

so

$$g^*(\lambda) = \frac{f^*(\lambda)}{(\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)} = \frac{f^*(\lambda)}{(\lambda - \lambda_1)(\lambda - \lambda_2)} = (\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

Let

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{f(\lambda) - f^*(\lambda)}{f(\lambda) + f^*(\lambda)}, \text{ so } \frac{1}{h(\lambda)}$$

$$= \frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)}.$$

In (9) above, since $\operatorname{Re} a_1 > 0$, $a > 0$ and $(1/h)^* = -1/h$, $1/h$ is a positive function (Theorem 5.2 of Levinson and Reddheffer[3]) and because g and g^* have no roots in common, g is a Hurwitz polynomial[3] (Theorem 5.1). Hence $\operatorname{Re} \lambda_3 < 0$ and and $\operatorname{Re} \lambda_4 < 0$. It remains to determine the nature of λ_1 and λ_2.

When $\lambda_1 = -\lambda_1$ and $\lambda_2 = -\bar{\lambda}_2$ then $\operatorname{Re} \lambda_1 = \lambda_2 = 0$.

In this case, the given inequality is easily verified as in the previous section with both λ_3 and λ_4.

Consider the second possibility where $\lambda_1 = -\bar{\lambda}_2$ or $\lambda_2 = -\bar{\lambda}_1$.

Here we have $\operatorname{Re} \lambda_1 = -\operatorname{Re} \lambda_2$ and $\operatorname{Im} \lambda_1 = \operatorname{Im} \lambda_2$.

since $\operatorname{Re} \lambda_4 < 0$ and $f(\lambda_4) = 0$, the inequality,

$4 \operatorname{Re} a_1' \operatorname{Re} (a_1' \bar{a}_2') - 3 (\operatorname{Im} a_2')^2 \geqslant 0$ where $a_1' = \lambda_4 + a_1$

$a_2' = \lambda_4^2 + a_1 \lambda_4 + a_2$ holds.

Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= \lambda^3 + a_1' \lambda^2 + a_2' \lambda + a_3'$$

where

$$a_3' = \lambda_4^3 + a_1 \lambda_4^2 + a_2 \lambda_4 + a_3 = -\frac{a_4}{\lambda_4}.$$

If we take up the argument towards the end of the converse of case 2 of Theorem 3.2, we can show that $\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = 0$.

Case 3— $f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - \lambda_4)$

$$g(\lambda) = \frac{f^*(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i(\lambda_3)} = \lambda - \lambda_4$$ is a Hurwitz polynomial for $\operatorname{Re} \lambda_4 < 0$.

$$g^*(\lambda) = -\frac{f^*(\lambda)}{(\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3))} = -(\lambda + \bar{\lambda}_4).$$

The function $h(\lambda) = \dfrac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}$ is positive (Lemma 3.1[4] and Theorem 5.1[3])

$h(\lambda)$ can also be put in the following forms :

$$h(\lambda) = \frac{\lambda^4 + i \operatorname{Im} a_1 \lambda^3 + \operatorname{Re} a_2 \lambda^2 + i \operatorname{Im} a_3 \lambda + \operatorname{Re} a_4}{\operatorname{Re} a_1 \lambda^3 + i \operatorname{Im} a_2 \lambda^2 + \operatorname{Re} a_3 \lambda + i \operatorname{Im} a_4}$$

(*equation continued on p.* 325)

$$= \frac{2\lambda + \bar{\lambda}_4 - \lambda_4}{-(\lambda_4 + \bar{\lambda}_4)} \cdot$$

Since h is positive, then (1) implies $\operatorname{Re} a_1 \neq 0$.

Because h appears as a first degree polynomial in λ, the remainder R' of the division of $\lambda^4 + i \operatorname{Im} a_1 \lambda^3 + \operatorname{Re} a_2 \lambda^2 + i \operatorname{Im} a_3 \lambda + \operatorname{Re} a_4$ by $\operatorname{Re} a_1 \lambda^3 + i \operatorname{Im} a_2 \lambda^2 + \operatorname{Re} a_3 \lambda + i \operatorname{Im} a_4$ mast be zero. That simply implies $a = r = 0$. and $\operatorname{Re} a_1 \operatorname{Re}(a_1 \bar{a}_4) - \operatorname{Im} a_2 \operatorname{Im} a_4 = 0$.

But

$$h(\lambda) = \frac{1}{\operatorname{Re} a_1} \lambda + \frac{i}{\operatorname{Re} a_1} (\operatorname{Im} a_1 - \frac{\operatorname{Im} a_2}{\operatorname{Re} a_1})$$

shows that $\operatorname{Re} a_1 > 0$, for h is a positive function.

Define

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)$$

$$= \lambda^3 + a'_1 \lambda^2 + a'_2 \lambda + a'_3$$

where

$$a'_1 = \lambda_4 = a_1,\ a'_2 = \lambda_4^2 + a_1 \lambda_4 + a_2,\ a'_3 = \lambda_4^3 + a_1 \lambda_4^2 + a_2 \lambda_4 + a_3$$

$$= -\frac{a_4}{\lambda_4}$$

with $K(\lambda)$, we are in the position of case 3, Theorem 3.2. Therefore the following must be true : $\operatorname{Re} a'_1 = \operatorname{Im} a'_2 \quad \operatorname{Re} a'_3 = 0$, and there exists a real λ_0 satisfying $K(i\lambda_0) = 0$, hence $f(i\lambda_0) = 0$ and $3\lambda_0^2 - 2i a'_2 \lambda_0 + a'^2_1 - 4a'_2 \leqslant 0$, this inequality is obtained by letting $\alpha = i\lambda_0$ in 3, Theorem 3.2.

Now $\operatorname{Re} a'_1 = \operatorname{Re}(\lambda_4 + a_1) = 0$, implies, $\operatorname{Re} \lambda_4 = -\operatorname{Re} a_1$, and

$$\operatorname{Im} a'_2 = 2 \operatorname{Re} \lambda_4 \operatorname{Im} \lambda_4 + \operatorname{Re} a_1 \operatorname{Im} \lambda_4 + \operatorname{Im} a_1 \operatorname{Re} \lambda_4 + \operatorname{Im} a_2 = 0$$

when we substitute $\operatorname{Re} \lambda_4 = -\operatorname{Re} a_1$ we get $\operatorname{Im} \lambda_4 = \frac{\operatorname{Im} a_2}{\operatorname{Re} a_1} - \operatorname{Im} a_1$.

Therefore

$$a'_1 = i \frac{\operatorname{Im} a_2}{\operatorname{Re} a_1}.$$

Also by substituting the values of Re λ_4 and Im λ_4 in a_2'

we get

$$a_2' = \frac{1}{(\text{Re } a_1)^2} [\text{Re } a_1 \text{ Re } (a_1 \bar{a}_2) - (\text{Im } a_2)^2]$$

and since

$$a = \text{Re } a_1 (\text{Re } a_1 - \bar{a}_2 - a_3) - (\text{Im } a_2)^2 = 0 \; a_2' = \text{Re } a_3/\text{Re } a_1.$$

Therefore λ_0 satisfies

$$3 \lambda_0^2 + 2 \frac{\text{Im } a_2}{\text{Re } a_1} \lambda_0 - \frac{(\text{Im } a_2)^2}{(\text{Re } a_1)^2} - 4 \frac{\text{Re } a_3}{\text{Re } a_1} \leqslant 0.$$

Conversely, suppose Re $a_1 > 0$, $a = r = 0$ and

Re a_1 Re $(a_1 \bar{a}_4)$ $-$ Im a_2 Im $a_4 = 0$ and assume the existence of real λ_0 with $f(i\lambda_0) = 0$ satisfying $3 \lambda_0 + 2 \frac{\text{Im } a_2}{\text{Re } a_1} \lambda_0 - \frac{(\text{Im } a_2)^2}{(\text{Re } a_1)^2} - 4 \frac{\text{Re } a_3}{\text{Re }\ _1} \leqslant 0.$

Take

$$\frac{f + f^*}{f - f^*} = \frac{\lambda^4 + i \text{ Im } a_1 \lambda^3 + \text{Re } a_2 \lambda^2 + i \text{ Im } a_3 \lambda + \text{Re } a_4}{\text{Re } a_1 \lambda^3 + i \text{ Im } a_2 \lambda^2 + \text{Re } a_3 \lambda + i \text{ Im } a_4},$$

the remainder of this division is zero since $a = r = 0$ and Re a_1 Re $(a_1 \bar{a}_4)$ $-$ Im a_2 Im $a_4 = 0$, as is easily checked.

Therefore $f + f^*$ and $f - f^*$ have three roots in common. By Lemma 3.2 f and f^* have three common roots λ_1, λ_2 and λ_3.

Let

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

and

$$f^*(\lambda) = (\lambda + \bar{\lambda}_1)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda - \bar{\lambda}_4).$$

It is obvious that λ_4 cannot be another common root to f and f^*, for then $f = f^*$ leading to Re $a_1 = 0$.

Since by assumption $i \lambda_0$ where λ_0 real, is a root of $f(\lambda)$, then it must be a common root to f and f^*. Therefore $i \lambda_0$ is different from λ_4. Suppose $\lambda_1 = i \lambda_0$.

$$f(\lambda) = (\lambda - i \lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

and

$$f^*(\lambda) = (\lambda - i \lambda_0)(\lambda + \bar{\lambda}_2)(\lambda + \bar{\lambda}_3)(\lambda + \bar{\lambda}_4).$$

If $\lambda_2 = -\bar{\lambda}_4$ then $\lambda_4 = -\bar{\lambda}_2$ and since $-\bar{\lambda}_2$ is a root of $f^*(\lambda)$, then λ_4 becomes a common root to f and f^* which is impossible. So $\lambda_2 \neq -\bar{\lambda}_4$ and similarly $\lambda_3 \neq -\bar{\lambda}_4$.

It is easy to show that we have only two possibilities :

either $\lambda_2 = -\bar{\lambda}_2$ leading to $\lambda_3 = -\bar{\lambda}_3$, or $\lambda_2 = -\bar{\lambda}_3$ which is equivalent to $\lambda_3 = \bar{\lambda}_2$.

In both cases $f^*(\lambda) = (\lambda - i\lambda_0)(\lambda - \lambda_2(\lambda - \lambda_3)(\lambda + \bar{\lambda}_4)$.

Let

$$g(\lambda) = \frac{f(\lambda)}{(\lambda - i\lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)} = (\lambda - \lambda_4)$$

then

$$g^*(\lambda) = -\frac{f^*(\lambda)}{(\lambda - i\lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)} = -(\lambda + \bar{\lambda}_4).$$

Define

$$h(\lambda) = -\frac{f(\lambda) + f^*(\lambda)}{f(\lambda) - f^*(\lambda)} = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)}.$$

Since $\text{Re}\, a_1 > 0$,

$$h(\lambda) = \frac{1}{\text{Re}\, a_1}\lambda + \frac{i}{\text{Re}\, a_1}\left(\text{Im}\, a_1 - \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\right)$$

is a positive function[3] (Theorem 5.2), and since g and g^* have no roots in common, g is a Hurwitz polynomial[3] (Theorem 5.1) implying that $\text{Re}\,\lambda_4 < 0$.

Also,

$$h(\lambda) = \frac{g(\lambda) - g^*(\lambda)}{g(\lambda) + g^*(\lambda)} = \frac{2\lambda + \bar{\lambda}_4 - \lambda_4}{-(\lambda_4 + \bar{\lambda}_4)} = -\frac{1}{\text{Re}\lambda_4}\lambda^2 + i\,\frac{\text{Im}\,\lambda_4}{\text{Re}\,\lambda_4}$$

Hence,

$$\text{Re}\,\lambda_4 = -\text{Re}\, a_1 \text{ and } \frac{\text{Im}\,\lambda_4}{\text{Re}\,\lambda_4} = \frac{1}{\text{Re} a_1}\left(\text{Im}\, a_1 - \frac{\text{Im}\, a_2}{\text{Re}\, a_1}\right)$$

leading to

$$\text{Im}\,\lambda_4 = \frac{\text{Im}\, a_2}{\text{Re}\, a_1} - \text{Im}\, a_1.$$

Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_4} = (\lambda - i\lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= \lambda^3 + a_2'\lambda^2 + a_2'\lambda + a_3'$$

where

$$a'_1 = \lambda_4 + a_1,\ a'_2 = \lambda_4^2 + a_1\lambda_4 + a_2,\ a'_3 = \lambda_4^3 + a_1\lambda_4^2 + a_2\lambda_4 + a_3$$

$$= -\frac{a_4}{\lambda_4}$$

With the above values of Re λ_4 and Im λ_4, and with the use of relation Re a_1 Re $(a_1 \bar{a}_4)$ − Im a_2 Im $a_4 = 0$, it is easy to show that Re a'_1 = Im a'_2 = Re a'_3 = 0. Let $\alpha = i\lambda_0$, then Re $\alpha = 0$ and $K(\alpha) = 0$. Now the given inequality takes the form :

$$3\alpha^2 + 2i\frac{\text{Im } a_2}{\text{Re } a_1}\alpha + \frac{(\text{Im } a_2)^2}{(\text{Re } a_1)^2} + 4\frac{\text{Re } a_3}{\text{Re } a_1} > 0.$$

But we know from the previous section that :

$a'_1 = i$ Im a_2/Re a_1, and a'_2 = Re a_3/Re a_1, therefore

$3\alpha^2 + 2a'_1\alpha - a'^2_1 + 4a'_2 > 0$. By 3 of Theorem 3.2, Re λ_2 = Re λ_3 = 0.

Case 4—$f(\lambda) = (\lambda - i\lambda_1)(\lambda - i\lambda_2)(\lambda - i\lambda_3)(\lambda - i\lambda_4)$, λ_1 is a possible zero real number, 2, 3 and 4 are non-zero real numbers.

Let

$$g(\lambda) = \frac{f(\lambda)}{\lambda - i\lambda_2} = (\lambda - i\lambda_1)(\lambda - i\lambda_3)(\lambda - i\lambda_4)$$

$$= \lambda^3 + a'_1\lambda^2 + a'_2\lambda + a'_3 \text{ where}$$

$$a'_1 = i\lambda_2 + a_1,\ a'_2 = -\lambda_2^2 + ia_1\lambda_2 + a_2 \text{ and}$$

$$a'_3 = -i\lambda_2^3 - a_1\lambda_2^2 + ia_2\lambda_2 + a_3 = -\frac{a_4}{i\lambda_2} = \frac{ia_4}{\lambda_2}$$

with $g(\lambda)$, we are in the position of case 3 of Theorem 3.2, therefore Re a'_1 = Im a'_2 = Re a'_3 = 0 which simply lead to

Re a_1 = Im a_2 = Re a_3 = Im a_4 = 0.

Theorem 3.2 also implies the existence of a number α, with Re $\alpha = 0$, $g(\alpha) = 0$, hence $f(\alpha) = 0$ and satisfying $3\alpha^2 + 2a'_1\alpha - a'^2_1\ 4a'_2 \geqslant 0$, obviously α coincides with either $i\lambda_1$, $i\lambda_3$ or $i\lambda_4$. If we let $\beta = i\lambda_2$, then Re $\beta = 0$, $f(\beta) = 0$ and $a'_1 = \beta + a_1$, $a'_2 = \beta^2 + a_1\beta + a_2$. Conversely, suppose Re a_1 = Im a_2 = Re a_3 = Im a_4 = 0, and there exist α and β such that Re α = Re β = 0,

$$f(\alpha) = f(\beta) = 0 \text{ satisfying } 3\alpha^2 + 2a_1' \alpha - a_1'^2 + 4a_2' \geqslant 0$$

where

$$a_1' = \beta + a_1, a_2' = \beta^2 + a_1 \beta + a_2.$$

Let

$$K(\lambda) = \frac{f(\lambda)}{\lambda - \beta} = \lambda^3 + a_1' \lambda^2 + a_2' \lambda + a_3' . \text{ So } K(\alpha) = 0,$$

and

$$\text{Re}\, a_1' = \text{Re}\, \beta + \text{Re}\, a_1 = 0, \text{ Im } a_2' = \text{Im } \beta^2 + a_1 \beta + a_2) = 0$$

$\text{Re}\, a_3' = \text{Re}\,(\beta^3 + a_1 \beta^2 + a_2 \beta + a_3) = 0.$ Now case 3 of Theorem 3.2 implies that all roots of $K(\lambda)$ must have zero real parts. That ends the Proof.

Finally, it is worth mentioning for the sake of applications that the arguments in all three theorems clarify the relations between the nature of the eigenvalues of the system and the coefficients of its characteristic equation.

References

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