Indian J. pure appl. Math., 20 (4): 330-333, April 1989

ON THE GRAPHOIDAL COVERING NUMBER OF A GRAPH

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(*Received* 31 *May* 1988)

A graphoidal cover of a graph $G = (V, E)$ is a collection ψ of (not necessarily open) paths in G such that (a) every path in ψ has at least two vertices (b) every vertex of G is an internal vertex of at most one path in ψ, and (c) every edge of G is in some path in ψ. The graphoidal covering number $\gamma(G)$ of G is defined to be the minimum cardinality of a graphoidal cover of G. In this paper we determine the graphoidal covering numbers of trees, complete bipartite graphs, Hamiltcnian graphs and regular graphs.

1. INTRODUCTION

By a graph we mean a finite undirected graph without loops or multiple edges. We follow the notation and terminology of Harary[2]. All graphs considered in this paper are assumed to be connected graphs without isolated points.

Let $G = (V, E)$ be a graph. We denote the number of vertices in G by p and the number of edges in G by q. If $P = (u_0, u_1, u_2, \ldots, u_n)$ is a path, not necessarily open, in G then u_0 and u_n are called terminals of P and $u_1, u_2, \ldots, u_{n-1}$ are called internal vertices of P. We denote by $t(P)$ the number of internal vertices of P. The following definition of graphoidal covering number of G is given in Devadas Acharya[1].

Definition—Let $G = (V, E)$ be a graph. A graphoidal cover of G is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

(1) Every path in ψ has atleast two vertices.

(2) Every vertex of G is an internal vertex of atmost one path in ψ.

(3) Every edges of G is in some path in ψ.

Let $\mathcal{G}(G)$ denote the set of all graphoidal covers of G. Then $\gamma(G) = \min_{\psi \in \mathcal{G}(G)} |\psi|$ is called the graphoidal covering number of G.

Thus $\gamma(G)$ is the minimum number of internally disjoint paths covering all the edges of G. We may further assume that every edge of G is in exactly one path of a graphoidal cover ψ so that $q = \sum_{P \in \psi} |E(P)| = |\psi| + \sum_{P \in \psi} t(P)$.

In this paper we obtain the graphoidal covering numbers of trees, complete bipartite graphs, Hamiltonian graphs and regular graphs.

Main Results

Theorem 1—$\gamma(G) = |E(G)|$ iff $G = K_2$.

Proof : If $G = K_2$ trivially $\gamma(G) = |E(G)| = 1$. Suppose $G \neq K_2$. Let P be a path in G such that $|E(P)| > 1$. Then $\psi = \{P\} \cup [E(G) - E(P)]$ is a graphoidal cover of G and $|\psi| < |E(G)|$ so that $\gamma(G) < |E(G)|$.

Theorem 2—Let G be a tree with n vertices of degree 1. Then $\gamma(G) = n - 1$.

Proof : We prove the result by induction on n. When $n = 2$, G is a path and hence $\gamma(G) = 1$. Suppose the result is true for any tree with $n - 1$ vertices of degree 1. Let G be a tree with n vertices of degree 1 where $n > 1$.

Let $P = (v_0, v_1, \ldots, v_k)$ be a path in G such that $d(v_0) > 2$, $d(v_k) = 1$ and $d(v_i) = 2$ for $i = 1, 2, \ldots, k - 1$.

Then $G_1 = G - \{v_1, v_2, \ldots, v_k\}$ is a tree with $n - 1$ vertices of degree 1. Hence there exists a graphoidal cover ψ of G_1 such that $|\psi| = n - 2$. Clearly $\psi \cup \{P\}$ is a graphoidal cover for G and hence $\gamma(G) \leqslant n - 1$.

Now suppose $P_1, P_2, \ldots, P_{n-2}$ is a graphoidal cover of G. Then $p - 1 = |E(G)| = (n - 2) + \sum_{i=1}^{n-2} t(P_i)$. However $\sum_{i=1}^{n-2} t(P_i) \leqslant p - n$ which gives a contradiction. Hence $\gamma(G) = n - 1$. This completes the induction and the proof.

Corollary 1—For any tree G, $\gamma(G) \geqslant \Delta - 1$ where Δ is the maximum degree of a vertex in G.

Corollary 2—Let G be a tree with $\Delta > 2$. Let v be a vertex in G such that $d(v) = \Delta$. Then $\gamma(G) = \Delta - 1$ if and only if $d(w) = 1$ or 2 for all vertices $w \neq v$.

Theorem 3—Let ψ be a graphoidal cover of G such that every vertex v with $d(v) > 1$ is an internal vertex of a path in ψ. Then $|\psi| = \gamma$.

Proof : Let ψ_1 be any graphoidal cover of G. Then $\sum_{P \in \psi_1} t(P) \leqslant \sum_{P \in \psi} t(P)$. Hence $q - |\psi_1| \leqslant q - |\psi|$ so that $|\psi_1| \geqslant |\psi|$. Hence $|\psi| = \gamma$.

Corollary—If there exists a graphoidal cover ψ of G such that every vertex of G is an internal vertex of a path in ψ then $\gamma(G) = q - p$.

PROOF : $q = \sum_{P \in \psi} |E(P)| = |\psi| + \sum_{P \in \psi} t(P) = |\psi| + p.$

Hence

$$\gamma(G) = |\psi| = q - P.$$

Theorem 4—Let G be a hamiltonian graph. Suppose there is a vertex v in G such that $d(v) > 3$. Then $\gamma(G) = q - p$.

PROOF : Let $C = (v = v_0, v_1, \ldots, v_{p-1}, v)$ be a Hamilton cycle in G. Since $d(v) > 3$, there exist vertices, say x and y, distinct from v_1 and v_{p-1} adjacent to v. Let $P = (x, v, y)$. Let S denote the set of all edges of G not covered by C and P. Then $\psi = \{C, P\} \cup S$ is a graphoidal cover for G and every vertex of G is an internal vertex of a path in ψ. Hence $\gamma(G) = q - p$.

Corollary 1—For the wheel $W_n = K_1 + C_{n-1}$ where $n \geqslant 5$, we have

$$\gamma(W_n) = q - p = n - 2.$$

Corollary 2—For the complete bipartite graph $K_{n,n}$ with $n > 3$, we have

$$\gamma(K_{n,n}) = q - p = n^2 - 2n.$$

Corollary 3—For $n > 4$, $\gamma(K_n) = q - p = \frac{1}{2} n(n - 3)$.

Theorem 5—Let G be a graph with $p > 5$. If G has a Hamilton path $P = (v_1, v_2, \ldots, v_p)$ such that v_1 and v_p have degrees $\geqslant 3$ in G, then $\gamma(G) = q - p$.

PROOF : Similar to that of Theorem 4.

We now determine $\gamma(G)$ for regular graphs.

Any 1-regular graph is K_2 and 2-regular graph is a cycle.

Hence $\gamma(G) = 1$ if G is 1-regular or 2-regular.

Theorem 6—Let G be a k-regular graph with $k > 3$. Then $\gamma(G) = q - p$.

PROOF : It is enough to construct a collection of mutually edge disjoint and internally disjoint paths in G such that every vertex of G is an internal vertex of a path in the collection.

Let $P_1 = (u_1, u_2, \ldots, u_n)$ be a longest path in G so that all vertices adjacent to u_1 or u_n are already in P_1. Since $k > 3$ we can find vertices x, y, z, w in P_1 such that x, y are distinct vertices each different from u_2 and are adjacent to u_1 and z, w are distinct vertices each different from u_{n-1} and are adjacent to u_n. If x, y, z, w, u_1 and u_n are all distinct, let $P_2 = (x, u_1, y)$ and $P_3 = (z, u_n, w)$. If one of x, y coincides with u_n and one of z, w coincides with u_1, say $x = u_n$ and $z = u_1$, let $P_2' = (y, u_1, u_n, w)$.

Thus we obtain a collection $\{P_1, P_2, P_3\}$ or $\{P_1, P_2'\}$ of edge disjoint and internally disjoint paths in which $u_1, u_2, \ldots, u_n$ are internal vertices of one path in the collection.

If these vertices exhaust all the vertices in G, the proof is complete. If not let w_1 be a vertex not lying on P_1 and let Q_1 be a longest path in G containing w_1 and internally disjoint with the paths constructed above. If the end points of Q_1 are not in P_1 we make them internal vertices of some path as before. We continue this process until all the vertices are exhausted and we obtain a required collection of paths.

We now proceed to determine the graphoidal covering number of complete bipartite graphs. $K_{1,1}$ is a path and $K_{2,2}$ is a cycle. Hence $\gamma(K_{1,1}) = \gamma(K_{2,2}) = 1$. It follows from Theorem 2 that for $n > 1$, $\gamma(K_{1,n}) = n - 1$. Also $\gamma(K_{2,3}) = 2$.

Theorem 7—Let G be the complete bipartite graph $K_{m,n}$ with $m > 2$ and $n > 2$ or $m = 2$ and $n > 3$. Then $\gamma(G) = q - p$.

PROOF : *Case i*—$m = 2$ and $n > 3$.

Let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $P_1 = (v_1, w_1, v_2, w_2, v_1)$, $P_2 = (v_2, w_3, v_1, w_4, v_2)$ and $P_{i-2} = (v_1, w_i, v_2)$ for $i = 5, 6, \ldots, n$.

Then $\psi = \{P_1, P_2, \ldots, P_{n-2}\}$ is a graphoidal cover of G and every vertex is an internal vertex of a path in ψ.

Case ii—$m = n$ and $n > 2$.

The result follows from Theorem 4.

Case iii—$m = n + 1$ and $n > 2$.

Let $X = \{v_1, v_2, \ldots, v_{n+1}\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G.

Let $P_1 = (v_1, w_1, v_2, w_2, \ldots, v_n, w_n, v_{n+1})$,

$P_2 = (w_2, v_{n+1}, w_3)$ and $P_3 = (w_2, v_1, w_3)$.

Every vertex of G is an internal vertex of one of these paths.

Case iv—$m > n + 2$ and $n > 2$.

Let $X = \{v_1, v_2, \ldots, v_m\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $P_1 = (v_1, w_1, v_2, w_2, \ldots, v_n, w_n, v_{n+1})$, $P_2 = (w_2, v_1, w_3)$ and $Q_i = (w_2, v_{n+i}, w_3)$ where $i = 1, 2, \ldots, m - n$. Every vertex is an internal vertex of one of these paths.

Hence $\gamma(G) = q - p$.

REFERENCES

1. B. Devados Acharya and E. Sampath Kumar, *Indian J. pure appl. Math.* **18** (1987), 882–90.
2. F. Harary, *Graph Theory*. Addison Wesley, 1972.