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ON α-HAUSDORFF SUBSETS, ALMOST CLOSED MAPPINGS AND ALMOST UPPER SEMICONTINUOUS DECOMPOSITION

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The purpose of the present paper is to study some properties of α-Hausdorff subsets, almost closed mappings and almost upper semicontinuous decomposition.

1. PRELIMINARIES

Our notation is standard. No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is 'regularly open' iff $\mathrm{IntCl}A = A$. A subset A of a space X is 'regularly closed' if $\mathrm{Int\ Cl}A = A$ (Singal and Singal[11]).

A subset A of a space X is α-paracompact (α-nearly paracompact) iff for every open (regularly open) cover $\mathcal{U}$ of A there exists an open X-locally finite family $\mathcal{B}$ which refines $\mathcal{U}$ and covers A (Kovačević[3] and Wine[12]).

A subset A of a space X is α-paracompact (α-nearly paracompact) with respect to a subset B of X iff for every open (regularly open) cover $\mathcal{U}=\{U_i : i \in I\}$ of A there exists an open family $\mathcal{V} = \{V_j : j \in J\}$ such that :

(i) $\mathcal{V}$ refines $\mathcal{U}$,

(ii) $A \subset U\{V_j : j \in J\}$,

(iii) $\mathcal{V}$ is locally finite at each point $x \in B$.

Subsets A and B of a space X are mutually α-paracompact (mutually α-nearly paracompact) iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and the subset B is α-paracompact (α-nearly paracompact) with respect to the subset A (Kovačević[4]).

A subset A of a space X is α-Hausdorff iff any two points a and b, such that $a \in A$ and $b \in X\setminus A$, can be strongly separated by open sets (Kovačević).

A subset A of space X is α-regular iff for any point $a \in A$ and any open subset U of X containing a there exists an open subset V such that $a \in V \subset \mathrm{Cl}V \subset U$ (Kovačević[6]).

A subset A of a space X is Lindelöf (nearly Lindelöf) iff every open (regularly open) cover of A has a countable subcover Kelly[1] and Kovačević.

A point p of a space X is a P-point iff $p \in \text{Int}\,(\cap \{U_n : n \in N\})$ whenever $\{U_n : n \in N\}$ is a sequence of neighbourhoods of p (Kunnen[7]).

A mapping $f : X \to Y$ is almost closed (almost open) iff for any regularly closed (regularly open) set F of X, $f(F)$ is closed (open) in Y (Singal and Singal[11]).

A decomposition $\mathscr{D}$ of a space X is upper semicontinuous (almost upper semicontinuous) iff for each D in $\mathscr{D}$ and each open (regularly open) set U containing D there exists an open set V such that $D \subset V \subset U$ and V is the union of members of $\mathscr{D}$ Kelly[1] and Kovačević.

2. On α-Hausdorff Subsets

Definition 2.1—A subset A of a space X is α-almost paracompact in respect to a subset B of X iff for every open cover $\mathscr{U} = \{U_i : i \in I\}$ of A there exists an open family $\mathscr{V} = \{V_j : j \in J\}$ such that :

(i) $\mathscr{V}$ refines $\mathscr{U}$,

(ii) $A \subset \text{Cl}\,(\cup \{V_j : j \in J\})$,

(iii) $\mathscr{V}$ is locally finite at each point $x \in B$.

Theorem 2.1—Let A be an α-Hausdorff α-almost paracompact subset with respect to each point of $X \setminus A$. Then A is closed.

Proof : Let A be any α-Hausdorff α-almost paracompact subset with respect to each point of $X \setminus A$. Let a be any point of $X \setminus A$. For each $x \in A$ there exist open sets U_x and V_x such that

$$x \in U_x,\ a \in V_x \text{ and } U_x \cap V_x = \phi.$$

Now

$$\mathscr{U} = \{U_x : x \in A\}$$

is an open covering of A. Since A is α-almost paracompact with respect to each point of $X \setminus A$, then exists an open family $\mathscr{V} = \{V_j : j \in J\}$ such that :

(i) $\mathscr{V}$ is locally finite at a,

(ii) $A \subset \text{Cl}\,(\cup \{V_j : j \in J\})$,

(iii) $\mathscr{V}$ refines $\mathscr{U}$.

Since $\mathscr{V}$ is locally finite at a there exists an open neighbourhood U of a and a finite subset J_0 of J such that

$$U \cap V_j \neq \phi \text{ for } j \in J_0 \text{ and } U \cap V_j = \phi \text{ for } j \in J \setminus J_0.$$

For each $j \in J_0$ there exists $x(j) \in A$ such that $V_j \subset U_{x(j)}$. Let

$$U_1 = U \cap (\cap \{V_{x(j)} : j \in J_0\}).$$

U_1 be an open neighbourhood of a such that

$$a \in U_1 \subset X \setminus A$$

hence, the subset $X \setminus A$ is open, i.e. the subset A is closed.

Theorem 2.2—Let A be any α-Hausdorff nearly Lindelöf subset of a space X. Let $a \in X \setminus A$ be a P-point. Then there are disjoint regularly open neighbourhoods of x and A.

Consequently, if each point of $X \setminus A$ is a P-point and A is an α-Hausdorff nearly Lindelöf subset of X, then A is closed.

PROOF: Since the subset A is α-Hausdorff, then for each point $x \in A$ there exist disjoint regularly open sets U_x and V_x such that

$$x \in U_x, a \in V_x.$$

Then

$$\mathcal{U} = \{U_x : x \in A\}$$

is a regularly open covering of A. Then there exists a countable subset A_0 of A such that

$$A \subset \cup \{U_x : x \in A_0\}.$$

Let

$$U = \cup \{U_0 : x \in A_x\} \text{ and } V = \text{Int} \cap \{V_x : x \in A_0\}.$$

Then U and V are open disjoint neighbourhoods of A and a respectively ($\alpha(U)$ and $\alpha(V) - \alpha(U) = \text{Int Cl}\, U$ — are regularly open neighbourhoods of A and a respectively).

Theorem 2.3—Let A and B be two disjoint α-Hausdorff nearly Lindelöf subsets of a space X such that each point of A and B is a P-point. Then there exist disjoint regularly open neighbourhoods of A and B respectively.

PROOF: For each point $x \in B$ there exist disjoint regularly open subsets U_x and V_x such that

$$x \in U_x \text{ and } A \subset V_x.$$

The family

$$\mathcal{U} = \{U_x : x \in B\}$$

is a regularly open covering of the subset B. Since B is nearly Lindelöf then there exists a countable subset B_0 of B such that

$$B \subset \cup \{U_x : x \in B_0\}.$$

Let

$$U = \alpha (\cup \{U_x : x \in B_0\}) \text{ and } V = \alpha(\text{Int} (\cap \{V_x : x \in B_0 \}).$$

Then U and V are disjoint regularly open neighbourhoods of B and A respectively.

Theorem 2.4—Let A and B be any disjoint α-Hausdorff subsets of a space X such that :

(i) each point of B is a P-point,

(ii) A is nearly Lindelöf,

(iii) B is α-nearly paracompact with respect to the subset A.

Then there are disjoint regularly open neighbourhoods of A and B respectively.

PROOF : For each point $x \in A$ there exist disjoint regularly open subsets U_x and V_x such that

$$x \in U_x \text{ and } B \subset V_x \text{ (Theorem 2.3 in Kovačević[4])}.$$

The family

$$\mathcal{U} = \{ U_x : x \in A\}$$

is a regularly open covering of A, hence there exists a countable subset A_0 of A such that

$$A \subset \cup \{U_x : x \in A_0\}.$$

Let

$$U = \alpha (\cup \{U_x : x \in A_0\}) \text{ and } V = (\text{Int} (\cap \{V_x : x \in A_0\}).$$

Then U and V are disjoint regularly open neighbourhoods of A and B respectively.

3. Almost Closed Mappings

Using the similar method as in Kovačević[4] we shall prove the following theorem.

· *Theorem* 3.1—Let X be a topological space such that each point of X is a P-point. Let $f : X \to Y$ be an almost closed mapping of the space X onto a space Y such that :

(i) for each point $y \in Y, f^{-1} (y)$ is an α-Hausdorff subset of X, ii for each point $y \in Y$, $f^{-1} (y)$ is nearly Lindelöf or α-nearly paracompact with respect to each subset $f^{-1} (z)$, $z \in Y$ and $z \neq y$.

Then Y is Hausdorff.

PROOF : Let y_1 and y_2 be any distinct points of Y. In any case, by hypothesis, there exist disjoint regularly open neighbourhoods U_1 and U_2 of $f^{-1}(y_1)$ and $f^{-1}(y_2)$ respectively. Since the mapping f is almost closed, then there exist open subsets of Y V_1 and V_2 containing y_1 and y_2 respectively such that

$$f^{-1}(y_1) \subset f^{-1}(V_1) \subset U_1 \,;\, f^{-1}(y_2) \subset f^{-1}(V_2) \subset U_2.$$

Hence the result.

Theorem 3.2—Let X be a topological space such that each point of X is a P-point. Let $f : X \to Y$ be an almost closed mapping of a space X onto a Lindelöf space Y such that :

(i) for each point $y \in Y$, $f^{-1}(y)$ is an α-Hausdorff subset of X,

(ii) for each point $y \in Y$ the subset $f^{-1}(y)$ is nearly Lindelöf or α-nearly paracompact with respect to each subset $f^{-1}(z)$, $z \in Y$ and $z \neq y$.

Then f is continuous.

PROOF : The proof is omitted. It is identical with the proof of Theorem 3.4 in Kovačević[4].

Theorem 3.3—If f is a closed and continuous mapping of a regular space X onto a space Y such that for each $y \in Y$ $f^{-1}(y)$ is an α-paracompact subset with respect to the subset $X \setminus f^{-1}(y)$, then Y is regular.

PROOF : Let $y \in Y$ and V be an open set containing y. Since the space X is regular and the subset $f^{-1}(y)$ is α-paracompact with respect to the subset $X \setminus f^{-1}(y)$, then, by Theorem 2.6 in Kovačević[4], there exists an open neighbourhood U of $f^{-1}(y)$ such that

$$f^{-1}(y) \subset U \subset \mathrm{Cl}U \subset f^{-1}(V).$$

Since f is closed, there exists an open set W in Y such that

$$y \in W \text{ and } f^{-1}(W) \subset U.$$

Therefore, we have

$$y \in W \subset f(U) \subset f(\mathrm{Cl}\, U) \subset V.$$

Hence

$$y \in W \subset \mathrm{Cl}\, W \subset V.$$

Hence the result.

Theorem 3.4—If f is an almost closed mapping of a regular space X onto a space Y such that for each point $y \in Y$ $f^{-1}(y)$ is an α-paracompact subset with respect to the subset $X \setminus f^{-1}(y)$, then f is closed.

PROOF : Let A be any closed subset of X and let $y \in X \setminus f(A)$. Since $f^{-1}(y) \subset X \setminus A$, there exists an open subset V such that

$$f^{-1}(y) \subset V \subset \mathrm{Cl}V \subset X \setminus A.$$

Then $\alpha(V)$ is a regularly open subset such that

$$f^{-1}(y) \subset \alpha(V) \subset \mathrm{Cl}\alpha(V) \subset X \setminus A.$$

Since f is almost closed, then there exists an open set W in Y such that

$$y \in W \text{ and } f^{-1}(y) \subset f^{-1}(W) \subset \alpha(V) \subset X \setminus A.$$

Therefore we have

$$y \in W \subset X \setminus f(A)$$

hence $Y \setminus f(A)$ is open. Then $f(A)$ is closed. Hence f is closed.

4. Almost Upper Semicontinuous Decomposition

Theorem 4.1—Let X be a topological space. Let $\mathscr{D}$ be an almost upper semicontinuous decomposition of X whose members are α-Hausdorff and mutually α-nearly paracompact subsets of X. Let $\mathscr{D}$ have a quotient topology. Then $\mathscr{D}$ is a Hausdorff space.

PROOF : The projection of the space X onto the space $\mathscr{D}$ is almost closed, hence, by Theorem 3.1 in Kovačević[4]. $\mathscr{D}$ is Hausdorff.

Theorem 4.2—Let X be a topological space such that each point of X is a P-point. Let $\mathscr{D}$ be an almost upper semicontinuous decomposition of X such that :

(i) each member of $\mathscr{D}$ is an α-Hausdorff subset of X,

(ii) each member $D \in \mathscr{D}$ is nearly Lindelöf or α-nearly paracompact with respect to each-member $V \in \mathscr{D}$, $D \neq V$.

Let $\mathscr{D}$ have the quotient topology. Then $\mathscr{D}$ is Hausdorff.

PROOF : The projection of the space X onto the space $\mathscr{D}$ is almost closed, hence $\mathscr{D}$ is Hausdorff.

Theorem 4.3— Let X be a regular topological space. Let $\mathscr{D}$ be an almost upper semicontinuous decomposition of the space X such that for each $D \in \mathscr{D}$, D is an α-paracompact subset with respect to the subset $X \setminus D$. Let $\mathscr{D}$ have the quotient topology. Then (i) $\mathscr{D}$ is upper semicontinuous, (ii) $\mathscr{T}$ is regular.

PROOF : The projection of the space X onto the space $\mathscr{D}$ is almost closed, hence the projection is closed. Hence $\mathscr{D}$ is an upper semicontinuous decomposition. Since the projection is closed then $\mathscr{D}$ is regular.

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