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CONVEX UNIVALENT POLYNOMIALS

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In this paper, we find a necessary condition and a sufficient condition for a biquadratic polynomial to be convex univalent.

1. INTRODUCTION

Let $f(z) = z + a_2 z^2 + \ldots$ be analytic in the unit disc $E = \{z : |z| < 1\}$. Let $F(z)$ be analytic and univalent in E. Let $f(0) = F(0)$. If the image of the unit disc E under the mapping f is contained in the image of the disc E by the mapping F, then $f(z)$ is said to be subordinate to $F(z)$ and this is denoted by $f(z) \prec F(z)$.

$f(z) = z + a_2 z^2 + \ldots$ is said to be convex univalent in E if and only if $\text{Re}\{1 + zf''(z)/f'(z)\} > 0$ in E.

$f(z) = z + a_2 z^2 + \ldots$ is said to be starlike univalent in E if and only if $\text{Re}\{zf'(z)/f(z)\} > 0$ in E.

The Hadamard product or convolution of 2 power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = z \neq \sum_{n=2}^{\infty} a_n b_n z^n$.

Frank[1,2] determined the necessary condition and sufficient condition for the polynomials $\sigma z + \mu a_2 z^2$ and $\sigma z + \mu a_2 z^2 + \beta a_3 z^3$ to be convex univalent. The aim of this paper is to determine the necessary condition and sufficient condition for the biquadratic polynomial $\sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ to be convex univalent and to show that

$$(1/2)\, z \prec V_3(z, f) \prec \sigma z + \mu a_2 z^2 + \alpha\beta a_3 z^3 + \gamma a_4 z^4 \prec f(z)$$

where $V_3(z, f)$ is the 3rd de la Vallee' Poussin mean of the function f given by the formula $V_3(z, f) = (3/10)\, z + (3/10)\, a_2 z^2 + (1/20)\, a_3 z^3$. In general, the nth de la Vallee' Poussin mean[3] of f is given by

$$V_n(z, f) = \binom{2n}{n}^{-1} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k, \quad (a_1 = 1).$$

2. Sufficient Condition

Theorem 1—If $f(z) = z + bz^2 + cz^3 + dz^4$ where b, c, d are real and non-negative, then the condition

$$\left(\frac{1 + 9c - 16d}{4}\right) \geqq b \geqq \left(\frac{8c - 30d - 14cd}{1 + 5c - 32d}\right)$$

ensures that $f(z)$ is convex.

Proof : It is enough to show that $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$ in E. Put $z = re^{i\theta}$, $0 \leqq r < 1$, $0 \leqq \theta \leqq 2\pi$. The above condition is equivalent to

$$\begin{aligned} 64d^2 r^6 + 84cd \cos\theta\, r^5 &+ (48bd \cos 2\theta + 27c^2) r^4 \\ &+ (20d \cos 3\theta + 30bc \cos\theta) r^3 + (12c \cos 2\theta + 8b^2) r^2 \\ &+ 6b \cos\theta\, r + 1 \geqq 0. \end{aligned} \qquad \text{...(2.1)}$$

If $r = 0$, obviously (2.1) holds. Therefore assume that $r \neq 0$. Denote the left hand side of (2.1) by $F(\theta)$. Then

$$\begin{aligned} F'(\theta) = -r \sin\theta\, \{84cd\, r^4 + 192bd \cos\theta\, r^3 &+ 30(6d - 8d \sin^2\theta + bc) r^2 \\ &+ 48c \cos\theta\, r + 6b\}. \end{aligned}$$

$$F'(\theta) = 0 \text{ gives } \sin\theta = 0$$

or

$$14cd\, r^4 - 32bd\, r^3 + (30d + 5bc) r^2 - 8c\, r + b = 0. \qquad \text{...(2.2)}$$

We shall show that

$$14cd\, r^4 - 32bd\, r^3 + (30d + 5bc) r^2 - 8c\, r + b \geqq 0. \qquad \text{...(2.3)}$$

Differentiate with respect to r and use the fact that $d \leqq 1/56$. The worst case happens when $r = 1$. This is equivalent to $14cd - 32bd + 30d + 5bc - 8c + b \geqq 0$ and this gives

$$b \geqq \frac{8c - 30d - 14cd}{1 + 5c - 32d}.$$

So when r lies in the open interval (0, 1), the inequality (2, 3) is satisfied.

It remains to show that when $F(\theta)$ takes its minimum value that (2.1) is satisfied. The value of $F(\theta)$, when $\theta = 0$, is seen to contain only positive terms for $r > 0$. Hence we have to consider (2.1) only when θ takes the value π. When $\theta = \pi$, (2.1) reduces to

$$(1 - 4br + 9cr^2 - 16dr^3)(1 - 2br + 3cr^2 - 4dr^3) \geqq 0. \qquad \text{...(2.4)}$$

The condition

$$b \geqq \frac{8c - 30d - 14cd}{1 + 5c - 32d}$$

implies that $b \geqq (3/4)\,(8c - 30d + 14cd)$. This, in turn, shows that $1 - 4br + 9cr^2 - 16dr^3 \leqq 1 - 2br + 3cr^2 - 4d^3, 0 < r < 1$. It is enough to show that $1 - 4br + 9cr^2 - 16dr^3$ has no root in the interval $(0, 1]$. Differentiate with respect to r and use $b \geqq (3/4)\,(8c - 30d + 14cd)$. This shows that the worst case happens when $r = 1$. This condition gives $1 \geqq 4b - 9r + 16d$. Therefore $b \leqq (1 + 9c - 16d)/4$. Hence the theorem is proved.

Corollary 1—Let

$$P_4(z) = \frac{3 + 14\gamma}{4} z + \frac{3 + 140\gamma}{10} z^2 + \frac{1 + 90\gamma}{20} z^3 + \gamma z^4$$

and $0 \leqq \gamma \leqq 1/70$. Then $P_4(z)$ is convex in E.

PROOF : $P_4(z)$ is convex if $\dfrac{4 P_4(z)}{3 + 14\gamma}$ is convex. That is

$$z + \frac{2(3 + 140\gamma)}{5(3 + 14\gamma)} z^2 + \frac{(1 + 90\gamma)}{5(3 + 14\gamma)} z^3 + \frac{4\gamma}{3 + 14\gamma} z^4$$

is convex.

The equalities

$$0 \leqq \frac{4\gamma}{3 + !4\gamma} \leqq \frac{1}{56}$$

and

$$\frac{2(3 + 140\gamma)}{5(3 + 14\gamma)} \geqq \frac{\dfrac{8 + 720\gamma}{5(3 + 14\gamma)} - \dfrac{120\gamma}{3 + 14\gamma} - \dfrac{56(1 + 90\gamma)\gamma}{5(3 + 14\gamma)^2}}{1 + \dfrac{1 + 90\gamma}{3 + 14\gamma} - \dfrac{128\gamma}{3 + 14\gamma}}$$

are equivalent to $0 \leqq \gamma \leqq (1/70)$.

Corollary 2—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be convex in E. Then so is

$$(P_4 * f)(z) = \frac{3 + 14\gamma}{4} z + \frac{3 + 140\gamma}{10} a_2 z^2 + \frac{1 + 90\gamma}{20}$$
$$\times\, a_3 z^3 + \gamma a_4 z^4.$$

This follows using the celebrated result of Ruscheweyh *et al.*[4].

Corollary 3—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be convex in E. Then

$$(P_n * f)(z) \prec V_n(z, f) \prec f(z)$$

and

$$(1/2)\, z \prec V_3(z, f) \prec (P_4 * f)(z) \prec V_4(z, f) \prec f(z).$$

PROOF : The relation $(P_n * f)(z) \prec V_n(z, f)$ follows from Frank[2] and Ruscheweyh *et al.*[4] by taking $f = P_n$ and $\varphi = f$ and $\psi = V_n\left(z, \frac{z}{1-2}\right)$. To obtain the second reation, we note that $(P_4 * f)(z) \prec V_4(z, f)$ follows from the above relation for $n = 4$. $V_3(z, f) \prec (P_4 * f)(z)$ follows from the fact that $(P_4 * f)(z)$ is convex and thus in Theorem 6 of Frank's paper[2], we take $\varphi = P_4$, $\psi = z/(1 - z)$ and $f = V(z, z/(1 - z)$. That $(1/2)\, z \prec V_3(z, f)$ follows from the fact that the image of the unit disc by a convex function covers the disc of radius 1/2 in the w-plane.

3. NECESSARY CONDITIONS

Theorem 2—If for all convex $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $\sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ is convex and

$$(1/2)\, z \prec V_3(f, z) \prec \sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4 \prec f(z)$$

then

$$\sigma = (1/2) + \mu - \beta + \gamma \text{ and } \mu \leqq (1/6) + (8/3)\,\beta - (17/3)\,\gamma.$$

PROOF : With $f(z) = z/(1 - z) = z + z^2 + z^3 + \ldots$ we can find the minimum value of $\sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ on $|z| = 1$ whence $|\sigma z + \mu z^2 + \beta z^3 + \gamma z^4| \geqq \sigma - \mu + \beta - \gamma$. If

$(1/2)\, z \prec \sigma z + \mu z^2 + \beta z^3 + \gamma z^4$, we must have $\sigma - \mu + \beta - \gamma \leqq (1/2)$. With the same $f(z) = z/(1 - z)$, if $\sigma z + \mu z^2 + \beta z^3 + \gamma z^4 \prec f(z)$ for real x, $-1 < x < 1$, we must have $\sigma x + \mu x^2 + \beta x^3 + \gamma x^4 \geqq -1/2$. Allow x to tend to -1. We get $\sigma \leqq (1/2) + \mu - \beta + \gamma$. Thus $\sigma = (1/2) + \mu - \beta + \gamma$. For real α, $-1 < \alpha < 1$, consider

$$F(z) = \frac{1 + \alpha^2}{1 - \alpha^2}\left[T(\alpha) - T\left(\frac{\alpha - z}{1 - \alpha z}\right)\right]$$

where $T(z) = \text{arc tan } z$. Then $F(z)$ is convex and maps E into the unit strip

$$\frac{1 + \alpha^2}{1 - \alpha^2}\{\text{arc tan } \alpha - \pi/4\} \leqq \text{Re } z \leqq \frac{1 + \alpha^2}{1 - \alpha^2}\{\text{arc tan } \alpha + \pi/4\}.$$

The function $F(z)$ has the Taylor expansion

$$F(z) = z + \sum_{n=2}^{\infty} \frac{\sin n\theta}{n \sin \theta} z^n$$

where

$$\sin \theta = \frac{1 - \alpha^2}{1 + \alpha^2} \text{ and } \cos \theta = \frac{2\alpha}{1 + \alpha^2} .$$

If the polynomial $\{(1/2) + \mu - \beta + \gamma\} z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4 \prec F(z)$, we must have for real x, $-1 < x < 1$

$$\arctan \alpha - \pi/4 \leqq \{1/2 + \mu - \beta + \gamma\} \sin \theta\, x + (\mu/2) \sin 2\theta\, x^2 + (\beta/3) \sin 3\theta\, x^3 + (\gamma/4) \sin 4\theta\, x^4.$$

But $\arctan \alpha - \pi/4 = -\theta/2$. Taking the limit as x tends to -1,

$$-\theta/2 \leqq -\{1/2 + \mu - \beta + \gamma\} \sin \theta + (\mu/2) \sin 2\theta - (\beta/3) \sin 3\theta + (\gamma/4) \sin 4\theta.$$

Taking the limit as α tends to 1 corresponds to θ tending to zero. By repeated application of L'Hospital's rule, result follows.

Theorem 3—If for all convex $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

$$\frac{3 + 14\gamma}{4} z + \frac{3 + 140\gamma}{10} a_2 z^2 + \frac{1 + 90\gamma}{20} a_3 z^3 + \gamma a_4 z^4$$

is convex and

$$(1/2) z \prec V_3(f, z) \prec \frac{3 + 14\gamma}{4} z + \frac{3 + 140\gamma}{10} a_2 z^2 + \frac{1 + 90\gamma}{20} a_3 z^3 + \gamma a_4 z^4 \prec f(z)$$

then $\gamma \leqq 1/70$.

Proof : Since $\sigma = 1/2 + \mu - \beta + \gamma$ when $\mu = 1/6 + (8/3)\beta - (17/3)\gamma$, the polynomial $f(z) = \sigma z + \mu a_2 z^2 + \beta a_3 z^3 + \gamma a_4 z^4$ becomes

$$\frac{3 + 14\gamma}{4} z + \frac{3 + 140\gamma}{10} a_2 z^2 + \frac{1 + 90\gamma}{20} a_3 z^3 + \gamma a_4 z^4.$$

The required result is a special case of Theorem 6 in Frank's paper[2] for $n = 4$. The proof is complete.

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