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MODIFIED MEANS

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(*In Memory of L. Shri L. S. Bosanquet*)

The object of the present paper is to determine a class of sequnces $\lambda = (\lambda_n)$ and absolute summability methods $|A|$ for which $|A|$ and its modified methods $|A', \lambda|$ are equivalent. The special case of Nörlund matrices is taken for special study.

1. Introduction and Notations

Let c and l denote the set of convergent and absolutely convergent sequences $x = (x_n)$. Given an infinite series $\sum_{n=0}^{\infty} a_n$ with (s_n) as the sequence of nth partial sums and an infinite matrix $A = (a_{nk})$, we write the sequence-to-sequence transformation $t = A(s)$ as

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k \qquad \ldots(1.1)$$

assuming that t_n exists for each $n \geqslant 0$. We write $\Phi_n = t_n - t_{n-1}$, $t_{-1} = 0$. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable (A) to the value s if $t \in c$ and $\sum_{n=0}^{\infty} \Phi_n = s$; and is said to be absolutely summable A if $\Phi \in l$ i. e. $\Sigma |\Phi_n| < \infty$. (Summation without limits means summation from 0 to ∞).

For a given sequence $\lambda = (\lambda_n)$ let τ_n be the sequence of A-transformation of $(\lambda_n a_n)$, that is,

$$\tau_n = \sum_{k=0}^{\infty} a_{nk} \lambda_k a_k. \qquad \ldots(1.2)$$

Suppose that $\lambda_n \neq 0$ whenever $n > 0$ and let

$$B = (b_{nk}) = \left(\frac{\lambda_k a_{nk}}{\lambda_n} \right) \qquad \ldots(1.3)$$

in the case $\lambda_0 = 0$, we define $a_{00} = b_{00}$. We write

$$\psi_n = \tau_n / \lambda_n = \sum_{k=0}^{\infty} b_{nk} a_k. \qquad \ldots(1.4)$$

The series Σa_n is said to be summable by the 'modified A-means of weight' λ or summable (A', λ) to the value s if $\Sigma \psi_n = s$; and absolutely summable by modified A means or summable $| A', \lambda |$ if $\psi \in 1$. In the case

$$\Sigma \Phi_n = \Sigma \psi_n$$

we say that the method A and $| A |$ are identical to their modified means with weight λ.

If the matrix is lower triangular, i. e. $a_{nk} = 0$ $(k > n)$, then (1.1) can be put in the form

$$t_n = \sum_{k=0}^{n} \bar{a}_{nk} a_k \qquad \ldots(1.5)$$

where

$$\bar{a}_{nk} = \sum_{\nu=k}^{n} a_{n\nu} \qquad \ldots(1.6)$$

and

$$\Phi_n = t_n - t_{n-1} = \sum_{k=0}^{n} \overset{\Delta}{a}_{nk} a_k \qquad \ldots(1.7)$$

where

$$\hat{A} = (\hat{a}_{nk}) \text{ and } \hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k}. \qquad \ldots(1.8)$$

Thus in the special case when

$$\hat{a}_{nk} = \frac{\lambda_k a_{nk}}{\lambda_n} = b_{nk} \qquad \ldots(1.9)$$

the method (A) is identical with (A', λ) mean.

Let (A_k^α) be the sequence of Cesàro coefficients determined by

$$\Sigma A_n^\alpha x^n = (1 - x)^{-\alpha-1} \ (| x | < 1).$$

If

$$a_{nk} = \begin{cases} \dfrac{A_{n-k}^{\alpha-1}}{A_n^\alpha} & (k \leqslant n) \\ 0 & (k > n) \end{cases} \qquad \ldots(1.10)$$

and $\alpha > -1$, $\lambda_n = n$, then (1.9) holds; for we know that

$$\frac{A^{\alpha}_{n-k}}{A^{\alpha}_{n}} - \frac{A^{\alpha}_{n-1-k}}{A^{\alpha}_{n-1}} = \frac{k}{n}\ A^{\alpha-1}_{n-k} \quad (\alpha > -1). \qquad \ldots(1.11)$$

Thus the Cesàro method (C, α) is identical with its modified mean.

The method (A) is said to be 'equivalent' to its modified mean with weight λ if

$$\Sigma\ \Phi_n \text{ converges} \Leftrightarrow \Sigma\ \psi_n \text{ converges}. \qquad \ldots(1.12)$$

Similarly $|\ A\ |$ is said to be equivalent to $|\ A', \lambda\ |$ if

$$\Phi \in 1 \Leftrightarrow \psi \in 1. \qquad \ldots(1.13)$$

When (1.12) [respectively (1.13)] holds we write $(A) \sim (A', \lambda)$, [respectively $|\ A\ | \sim |\ A', \lambda\ |$]. We write

$$\theta\,(A) = \{\lambda \in \mathrm{R} : (A) \sim (A', \lambda)\}. \qquad \ldots(1.14)$$

Let N be a Nörlund martix (N, p) defined by

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}}{P_n} & (k \leqslant n) \\ 0 & (k > n) \end{cases} \qquad \ldots(1.15)$$

where $P_n = p_0 + p_1 + \ldots + p_n \neq 0$ for $n \geqslant 0$. Let (N', p, λ) denote the modified Nörlund mean.

When $p = (p_n)$ satisfies Kaluza condition :

$$p_n > 0,\ p_{n+1}\ p_{n-1} \geqslant p_n^2\ ,\ p_{n+1} \leqslant p_n \qquad \ldots(1.16)$$

we write $p \in \mathscr{M}$. Das[4] has shown that when $p \in \mathscr{M}$,

$$n \in \theta\,(\ |\ N\ |\ ) \text{ i. e. } |\ N, p\ | \sim |\ N', p, \lambda\ | \text{ with } \lambda_n = n. \qquad \ldots(1.17)$$

Bosanquet and Das[1] have shown that a Nörlund method is identical wth its modified Nörlund mean of weight $\lambda_n = n$ if and only if the Nörlund matrix is a Cesàro matrix.

It is easy to verify that Riesz method $(R, \mu_n, 1)$ is identical to its modified mean with weight

$$\lambda_n = \frac{\mu_n}{\mu_{n+1} - \mu_n}\ .$$

When a method is identical with its modified mean as in the case of Cesàro or Hausdorff mean [see Hardy[5], p. 247] it is an ideal situation; because the modified means are usually simpler to work with. Even when identification fails, equivalence may sometimes hold and this serves as well. There are again Nörlund methods which are not equivalent to their modified means[1].

The knowledge of the scope of $\theta(A)$ may be helpful in dealing with the summability factor problem: Suppose that we are required to determine if $\Sigma \epsilon_n a_n$ is summable A (or $|A|$). If we know that $\lambda \in \theta(A)$, then instead of considering A-mean of $\Sigma \epsilon_n a_n$, we may consider

$$(\alpha_n) = \left(\frac{1}{\lambda_n} \Sigma a_{nk} \epsilon_k \lambda_k a_k \right).$$

Now we may choose (ϵ_n), or $(\lambda_n) \in \theta(A)$, or both, in such a way that

$$\epsilon_n \lambda_n = n \text{ or } \epsilon_n \lambda_n = 1.$$

In that case α_n reduces respectively to

$$\alpha'_n = \frac{\epsilon_n}{n} \Sigma a_{nk} k a_k$$

and

$$\alpha''_n = \epsilon_n \Sigma a_{nk} a_k$$

and obviously α'_n, α''_n are simpler to work with as the factor ϵ_n is outside the sigmas.

We need some additional definitions and notations.

The matrix $A = (a_{nk})$ is said to be normal if it is lower triangular with non-zero diagonal elements. It may be noted that the normal matrix A has a two sided normal inverse which is denoted as $A^{-1} = \left(a^{-1}_{nk} \right)$.

We write

$$\theta_{m\nu} = \sum_{k=\nu}^{m} a^{-1}_{k\nu}, \quad \theta^*_{m\nu} = \sum_{k=\nu}^{m} | a^{-1}_{k\nu} | \qquad \text{...(1.18)}$$

$$\theta_\nu = \sum_{k=\nu}^{\infty} a^{-1}_{k\nu}, \quad \theta^*_\nu = \sum_{k=\nu}^{\infty} | a^{-1}_{k\nu} | \qquad \text{...(1.19)}$$

whenever these exist.

We write

$$\Omega(n, k) = \begin{cases} \dfrac{P_{n-k}}{P_n} - \dfrac{P_{n-k-1}}{P_{n-1}}, & \text{for } k \leqslant n \\ 0, & \text{for } k > n. \end{cases} \qquad \text{...(1.20)}$$

Let the sequence (c_n) be defined formally by

$$(\Sigma\, c_n\, x^n)\,(\Sigma\, p_n\, x^n) = 1 \qquad \dots(1.21)$$

where $p_0 \neq 0$, that is, by equations

$$\sum_{k=0}^{n} p_{n-k}\, c_k = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0) \end{cases} \qquad \dots(1.22)$$

and let

$$c_{-1} = 0.$$

We write, for any sequence (f_n)

$$\Delta^0 f_n = f_n,\ \Delta f_n = f_n - f_{n+1},\ \Delta^h f_n = \Delta\,(\Delta^{h-1} f_n)$$

$$\nabla^0 f_n = f_n,\ \nabla f_n = f_n - f_{n-1},\ \nabla^h f_n = \nabla\,(\nabla^{h-1} f_n)$$

for $h = 1, 2, 3, \dots$, and define $\nabla^{-1} f_n = f_0 + f_1 + \dots + f_n$. We adopt the convention that $f_n = 0$ if $n < 0$. Further we write

$$f_n^{(0)} = f_n,\ f_n^{(h)} = f_0^{(h-1)} + f_1^{(h-1)} + \dots + f_n^{(h-1)}$$

for $h = 1, 2, \dots$.

In Section 2 we make an attempt to discuss about the general mean $|\,A', \lambda\,|$, whereas in Section 3 and onward we confine ourselves only to modified Nörlund mean $|\,N', p, \lambda\,|$. The question of modified (A', λ) mean has however not been discussed in the paper.

2. Matrix Methods

We first obtain the following fundamental lemma.

Lemma 1—Let $A = (a_{nk})$ be a triangular matrix and let $\hat{A}$ and B be defined by (1.3) and (1.8). Suppose also that $\hat{A}^{-1}$ and B^{-1}, the inverses of $\hat{A}$ and B respectively, exist. Let

$$\hat{A}\, B^{-1} = \left(\sum_{k=r}^{n} \hat{a}_{nk}\; b_{kr}^{-1} \right) = G = (g_{n,r})$$

and

$$B\, \hat{A}^{-1} = \left(\sum_{k=r}^{n} b_{nk}\, \hat{a}_{kr}^{-1} \right) = H = (h_{n,r}).$$

(a) In order that $| A', \lambda | \Rightarrow | A |$ it is necessary and sufficient that, for all k,

$$\sum_{n=k}^{\infty} | g_{nk} | \leqslant K \qquad \text{.. (2.1)}$$

where K is an absolute positive constant not necessarily the same at each occurrence.

(b) In order that $| A | \Rightarrow | A', \lambda |$, it is necessary and sufficient that, for all k,

$$\sum_{n=k}^{\infty} | h_{n,k} | \leqslant K.$$

PROOF : We write (1.4) and (1.7) as

$$\psi = B\,(a), \;\; \Phi = \hat{A}\;(a). \qquad \text{...(2.2)}$$

Since the inverses B^{-1}, $\hat{A}^{-1}$ exist, we obtain

$$a = B^{-1}\,(\psi), \; a = \hat{A}^{-1}\,(\Phi).$$

Hence

$$\Phi = \hat{A}\; B^{-1}\,(\psi) = G\,(\psi) \qquad \text{...(2.3)}$$

and

$$\psi = B\,\hat{A}^{-1}\,(\Phi) = H\,(\Phi). \qquad \text{...(2.4)}$$

It follows from a result of Knopp and Lorentz[6] and (2.3) that

$$\psi \in 1 \Rightarrow \Phi \in 1 \text{ (i. e. } G : 1 \rightarrow 1)$$

if and only if (2.1) holds. Similarly (2.4) yields $\Phi \in 1 \Rightarrow \psi \in 1$ if and only if (2.2) holds. This completes the proof.

Now we prove

Theorem 1—Let A be a normal matrix and (λ) be a positive and non-decreasing sequence if

$$\hat{A}\, A^{-1} : 1 \rightarrow 1\,; \qquad \text{...(2.5)}$$

$$\hat{A} \; : 1 \rightarrow 1 \qquad \text{...(2.6)}$$

$$\sum_{n>k} \frac{\lambda_n - \lambda_k}{\lambda_n} \;|\; \Delta_k \hat{a}_{nk} \;| = O\left(\frac{\lambda_{k+1} - \lambda_k}{\lambda_{k+1}} \right); \qquad \text{...(2.7)}$$

and

$$\sum_{n=k}^{\infty} |\theta_{nk}| \Delta \left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{\lambda_k}\right) \qquad \ldots(2.8)$$

where (θ_{nk}) is defined in (1.18). Then

$$|A, \lambda| \Rightarrow |A|.$$

Remarks : (i) Note that the condition (2.5) means that if $x \in 1$, then $\hat{A}\, A^{-1} x \in 1$, and then this is so if and only if

$$\sum_{n=\mu}^{\infty} \left| \sum_{\nu=\mu}^{n} \hat{a}_{n\nu}\, a^{-1}_{\nu\mu} \right| \leqslant K, \text{ for } \mu = 0, 1, 2, \ldots \text{ (See Knopp and Lorentz[6]).} \qquad \ldots(2.9)$$

Similarly (2.6) holds if and only if

$$\sum_{n=\mu}^{\infty} |\hat{a}_{n\mu}| \leqslant K \qquad \ldots(2.10)$$

for $\mu = 0, 1, 2, \ldots$.

(ii) In Lemma 3, we have discussed some simple conditions under which (2.8) holds good.

PROOF : To prove the theorem, we have to prove (by Lemma 1 (a)) that, for $\mu = 0, 1, 2, \ldots$

$$J_\mu = \sum_{n=\mu}^{\infty} |h_{n\mu}| = \sum_{n=\mu}^{\infty} \left| \sum_{\nu=\mu}^{n} \hat{a}_{n\nu}\, b^{-1}_{\nu\mu} \right| \leqslant K.$$

Since

$$b^{-1}_{\nu\mu} = \frac{\lambda_\mu}{\lambda_\nu}\, a^{-1}_{\nu\mu}$$

$$= \frac{\lambda_\mu}{\lambda_n}\, a^{-1}_{\nu\mu} + \frac{(\lambda_n - \lambda_\nu)\,\lambda_\mu}{\lambda_\nu\, \lambda_n}\, a^{-1}_{\nu\mu}, \quad \mu \leqslant \nu \leqslant n$$

we obtain

$$J_\mu \leqslant J^{(1)}_\mu + J^{(2)}_\mu$$

where, by hypothesis (2.5) and since $\lambda_n > 0$ and non-decreasing

$$J^{(1)}_\mu = \lambda_\mu \sum_{n=\mu}^{\infty} \frac{1}{\lambda_n} \left| \sum_{\nu=\mu}^{\infty} \hat{a}_{n\nu}\, a^{-1}_{\nu\mu} \right|$$

(*equation continued on p.* 354)

$$< \sum_{n=\mu}^{\infty} \left| \sum_{\nu=\mu}^{n} \hat{a}_{n\nu} \; a^{-1}_{\nu\mu} \right| \leqslant K$$

for $\mu = 0, 1, 2, \ldots.$

$$J^{(2)}_{\mu} = \lambda_{\mu} \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \frac{\lambda n - \lambda_\nu}{\lambda_\nu} \hat{a}_{n\nu} \; a^{-1}_{\nu\mu} \right|$$

$$= \lambda_{\mu} \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \Delta_\nu \left(\frac{\lambda_\nu - \lambda_\nu}{\lambda_\nu} \hat{a}_{n\nu} \right) \theta_{\nu\mu} \right|$$

$$\leqslant \lambda_{\mu} \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \lambda n \, \Delta \, (1/\lambda_\nu) \, \hat{a}_{n\nu+1} \, \theta_{\nu\mu} \right|$$

$$+ \lambda_{\mu} \sum_{n=\mu}^{\infty} \frac{1}{\lambda n} \left| \sum_{\nu=\mu}^{n} \frac{\lambda n - \lambda_\nu}{\lambda_\nu} (\Delta_\nu \, \hat{a}_{n\nu}) \, \theta_{\nu\mu} \right|$$

$$= J^{(21)}_{\mu} + J^{(22)}_{\mu}, \text{ say.}$$

Now by (2.6) and (2.8) we obtain

$$J^{(21)}_{\mu} \leqslant \lambda_{\mu} \sum_{\nu=\mu}^{\infty} | \theta_{\nu\mu} | \, \Delta \left(\frac{1}{\lambda_\nu} \right) \sum_{n=\nu}^{\infty} | \hat{a}_{n,\nu+1} |$$

$$\leqslant K \; \lambda_{\mu} \sum_{\nu=\mu}^{\infty} | \theta_{\nu\mu} | \, \Delta \left(\frac{1}{\lambda_\nu} \right)$$

$$\leqslant K, \text{ for } \mu = 0, 1, 2, \ldots$$

and by (2.7) and (2.8)

$$J^{(22)}_{\mu} \leqslant \lambda_{\mu} \sum_{\nu=\mu}^{\infty} | \theta_{\nu\mu} | \; \frac{1}{\lambda_\nu} \sum_{n=\nu}^{\infty} \left| \frac{\lambda n - \lambda_\nu}{\lambda n} \Delta_\nu \, \hat{a}_{n\nu} \right|$$

$$\leqslant K \; \lambda_{\mu} \sum_{\nu=\mu}^{\infty} | \theta_{\nu\mu} | \; \Delta \left(\frac{1}{\lambda_\nu} \right) \leqslant K$$

for $\mu = 0, 1, 2, \ldots.$

This completes the proof of Theorem 1.

Remark : It follows from Lemma 1 (b) that a necessary and sufficient condition for $| A | \Rightarrow | A', \lambda |$ is that (2.2) should hold, i.e.

$$\sum_{n=\mu}^{\infty} \left| \sum_{\nu=\mu}^{n} b_{n\nu} \overset{\vee}{a}{}^{-1}_{\nu\mu} \right| \leqslant K, \mu = 0, 1, 2, \ldots.$$

Clearly (2.2) holds if

$$\text{(i)} \quad \sum_{n=k}^{\infty} | b_{nk} | \leqslant K, \; k = 0, 1, 2, \ldots$$

and

$$\text{(ii)} \quad \sum_{\nu=\mu}^{\infty} | \hat{a}^{-1}_{\nu\mu} | \leqslant K, \; \mu = 0, 1, 2, \ldots.$$

However the above condition (ii) does not hold in the Cesàro case. For in that case

$$\hat{a}^{-1}_{nk} = \sum_{r=k}^{n} A_r^{\alpha} \; A_{n-r}^{-\alpha-2} \; .$$

It may be remarked that so far it has not been possible for the author to determine some suitable simple conditions on a general matrix A in order that the condition (2.2) for $| A | \Rightarrow | A', \lambda |$ be satisfied so as to include both the Cesàro and the general Nörlund case. However the problem of establishing $| N', p, \lambda | \sim | N, p |$ has been tackled in a somewhat satisfactory manner in later sections.

An analysis of Conditions

We now make an analysis of conditions in Theorem 1. For this we make some preparation.

Let

$$d_{nk} = \frac{a_{n+1,k}}{a_{n,k}} \quad (0 \leqslant k \leqslant n, n = 0, 1, 2, \ldots).$$

We write $A \in \mathfrak{M}^*$ if

A is normal, $a_{nk} > 0$ for $k \leqslant n$ ...(2.11)

$d_{nk} \leqslant d_{n,k-1}$ for $0 \leqslant k \leqslant n$, ...(2.12)

$d_{nk} \leqslant 1$ for $0 \leqslant k \leqslant n$. ...(2.13)

The following results are known.

Lemma 2—(a) Let A satisfy conditions (2.11) and (2.12). Then its normal inverse satisfies

$$a_{nk}^{-1} \leqslant 0 \ (k < n), \ a_{nn}^{-1} = \frac{1}{a_{nn}} > 0. \qquad \ldots(2.14)$$

(b) If A satisfies conditions (2.11) and (2 13), then

$$\sum_{k=r}^{n} a_{nk}^{-1} \geqslant 0, \ 0 \leqslant r \leqslant n, \ n = 0,1,2, \ldots \qquad \ldots(2.15)$$

(c) Let $A \in \mathscr{M}^*$. Then

$$\sum_{k=0}^{n} | \ a_{nk}^{-1} \ | \leqslant 2/a_{nn} \qquad \ldots(2.16)$$

(d) If in addition, $\sum_{k=0}^{n} a_{nk}$ is non-decreasing as n increases, then

$$| \ a_{nk}^{-1} \ | \leqslant | \ a_{k+1,k}^{-1} \ | \ (n \geqslant k+1). \qquad \ldots(2.17)$$

See Peyerimhoff[8] (p. 33) for the result (2.14). For the result (2.15), see Peyerimhoff[7] (Satz[4]). The result (2.18) is a trivial deduction from (2.14) and (2.15). The result (2.17) is due to Russell[9].

Remark : The result (2.15) fails to hold under the hypotheses (2.11) and (2.12). For example, let (a_n) be a sequence of positive numbers and define

$$a_{nk} = a_n \ (0 \leqslant k \leqslant n).$$

Then the hypotheses (2.11) and (2.12) are satisfied. But if (t_n) is the A-transform of (s_n), then

$$t_n = a_n \sum_{k=0}^{n} a_k$$

so that

$$s_n = -\frac{t_{n-1}}{a_{n-1}} + \frac{t_n}{a_n}.$$

Thus

$$a_{nk}^{-1} = \begin{cases} -\dfrac{1}{a_{n-1}} & (k = n-1) \\ \dfrac{1}{a_n} & (k = n) \\ 0 & \text{(otherwise).} \end{cases}$$

Hence for a given $n \geqslant 1$, (2.15) holds for all $k \leqslant n$ if and only if

$$a_{n-1} \geqslant a_n.$$

In the following theorem we examine the case for d_{nk} a constant.

Theorem 2—Let A be normal and let $a_{nk} \neq O$ $(k \leqslant n)$.

Then

$$d_{nk} = \alpha\ (\neq 0),\ (k \leqslant n) \qquad \ldots(2.18)$$

if and only if

$$\left.\begin{aligned} a^{-1}_{n\nu} &= O\ (n \geqslant \nu + 2) \\ a_{k+1,\,k+1}\ a^{-1}_{k+1,k} &= -\ \alpha. \end{aligned}\right\} \qquad \ldots(2.19)$$

On the otherhand

and

$$\left.\begin{aligned} a^{-1}_{nk} &= O\ (k + 2 \leqslant n \leqslant N + 1) \\ a^{-1}_{N+2,k} &\neq 0 \end{aligned}\right\} \qquad \ldots(2.20)$$

if and only if

$$\left.\begin{aligned} d_{nk} &= \alpha\ (k \leqslant n \leqslant N) \\ d_{N+1,k} &\neq d_{N+1,k+1}. \end{aligned}\right\} \qquad \ldots(2.21)$$

Proof : Suppose that (2.19) holds. Since

$$a^{-1}_{k+1,k}\ \ a_{kk} + a^{-1}_{k+1,k+1}\ \ a_{k+1,k} = 0 \qquad \ldots(2.22)$$

we have

$$d_{kk} = \frac{a_{k+1,k}}{a_{kk}} = -\ a_{k+1,k+1}\ a^{-1}_{k+1,k} = \alpha.$$

Also we have (see Russell[9], p. 101)

$$a_{n+1,n+1}\ a^{-1}_{n+1,k} = \sum_{\nu=k+1}^{n} a_{n\nu}\ (d_{nk} - d_{n\nu})\ a^{-1}_{\nu k}\ . \qquad \ldots(2.23)$$

Since $a^{-1}_{n+1,k} = O\ (n > k + 1)$, (2.23) reduces to :

$$a_{n,k+1}\ (d_{nk} - d_{n,k+1})\ a^{-1}_{k+1,k} = O\ (n \geqslant k + 1) \qquad \ldots(2.24)$$

and since for $n \geqslant k + 1$, $a_{n,k+1} \neq 0$, $a^{-1}_{k+1,k} \neq 0$, we obtain from (2.24) that

$$d_{nk} = d_{n,k+1} \ (n \geqslant k+1).$$

Hence (2.18) holds.

If, on the contrary, (2.18) holds, then

$$a_{nk} = \alpha^{n-k} a_{kk}$$

and from the formula (2.22) and (2.23) we can reduce (2.19).

If (2.20) holds, we obtain

$$a_{N+2,N+2} \, a^{-1}_{N+2,k} = a_{N+1,N+1} \ (d_{N+1,k} - d_{N+1,k+1}) \, a^{-1}_{k+1,k}$$

and from this we obtain (2.21). The proof that (2.21) implies (2.20) follows from (2.23).

This completes the proof.

We write $A \in \mathscr{M}^{**}$ if A satisfies conditions (2.11), (2.12) and the following :

$$a_{nk} \leqslant a_{nk+1} \ (0 \leqslant k \leqslant n-1). \qquad ...(2.25)$$

Now we prove :

Theorem 3—Let $A \in \mathscr{M}^{**}$. Then for all m, n and r, $m \geqslant n \geqslant r$,

(i) $\theta_{mn} \geqslant \theta_{m+1,n} \geqslant 0; \ \theta_n \geqslant 0;$

(ii) $a_{nn} \ (\theta^*_n + \theta_n) = 2;$

(iii) $0 \leqslant \sum_{k=\nu}^{r} a_{nk} \, a^{-1}_{k\nu} \leqslant a_{n\nu} \, \theta_{r\nu} \ (0 \leqslant \nu < r \leqslant n);$

(iv) $\sum_{k=n}^{m} a_{kn} \sum_{\nu=m+1}^{\infty} | \, a^{-1}_{\nu n} \, | \leqslant 1.$

Proof : Let $r < n$. Since $a_{nk} \leqslant a_{n,k+1}$ and $a^{-1}_{k\nu} \leqslant 0 \ (k \geqslant \nu+1)$ by Lemma 2 (a), we obtain :

$$\sum_{k=\nu}^{r} a_{nk} \, a^{-1}_{k\nu} = a^{-1}_{\nu\nu} \, a_{n\nu} - | \, a^{-1}_{\nu+1,\nu} \, | \, a_{n,\nu+1} - \ldots - | \, a^{-1}_{r\nu} \, | \, a_{nr}$$

$$= a_{n\nu} \left(a^{-1}_{\nu\nu} - | \, a^{-1}_{\nu+1,\nu} \, | \, \frac{a_{n,\nu+1}}{a_{n\nu}} - \ldots - | \, a^{-1}_{r\nu} \, | \, \frac{a_{nr}}{a_{n\nu}} \right)$$

$$\leqslant a_{n\nu} \left(a^{-1}_{\nu\nu} - | \, a^{-1}_{\nu+1,\nu} \, | - \ldots - | \, a^{-1}_{r\nu} \, | \right) = a_{n\nu} \, \theta_{r,\nu}. \qquad ...(2.26)$$

On the otherhand

$$\sum_{k=\nu}^{r} a_{nk}\, a_{k\nu}^{-1} \geqslant \sum_{k=\nu}^{n} a_{nk}\, a_{k\nu}^{-1} = \delta_{n\nu} = \begin{cases} 1 \ (n = \nu) \\ 0 \ (n \neq \nu) \end{cases} .$$

Hence from (2.26) we obtain (iii).

Now from (iii) we obtain $\theta_{mn} > 0$ and by making $m \to \infty$, we find that θ_n exists and that $\theta_n \geqslant 0$. Also $\theta_{mn} \geqslant \theta_{m+1,n}$ is trivially true after Lemma 2(a).

Since for $n \geqslant \nu$,

$$\sum_{k=\nu}^{n} | a_{k\nu}^{-1} | + \sum_{k=\nu}^{n} a_{k\nu}^{-1} = 2a_{\nu\nu}^{-1} > 0$$

we obtain, by making $n \to \infty$,

$$\left(\theta_{\nu}^{*} + \theta_{\nu} \right) = \frac{2}{a_{\nu\nu}} .$$

Hence (ii) follows. Again

$$0 \leqslant \theta_{\nu} = \left(\sum_{k=\nu}^{m} + \sum_{k=m+1}^{\infty} \right) a_{k\nu}^{'-1} = \theta_{m\nu} - \sum_{k=m+1}^{\infty} | a_{k\nu}^{-1} |$$

so that

$$\sum_{k=m+1}^{\infty} | a_{k\nu}^{-1} | \leqslant \theta_{m\nu}.$$

Hence

$$\sum_{k=n}^{m} a_{kn} \sum_{r=m+1}^{\infty} | a_{rk}^{-1} | \leqslant \sum_{k=n}^{m} a_{kn}\, \theta_{mk} = 1.$$

This completes the proof of Theorem 3.

As commented earlier, we now give the following result before we take up the study of the problem for Nörlund means.

Lemma 3—Let $A \in \mathscr{M}^{**}$ and let $\lambda_n > 0$, non-decreasing such that

$$n\,(\lambda_{n+1} - \lambda_n) = O\,(\lambda_n). \qquad ...(2.27)$$

Also let

$$\sum_{\nu=\mu}^{2\mu} \theta_{\nu\mu} = O(\mu). \qquad \ldots(2.28)$$

Then (2.8) holds.

PROOF : By Theorem 3 (i)

$$0 \leqslant \theta_{2\mu,\mu} < \frac{1}{\mu} \sum_{\nu=\mu}^{2\mu} \theta_{\nu\mu}$$

and therefore (2.28) implies that

$$\theta_{2\mu,\mu} = O(1).$$

Now by Theorem 3 (i)

$$\sum_{k=\mu}^{\infty} \theta_{k\mu} \Delta\left(\frac{1}{\lambda_k}\right) = \left(\sum_{k=\mu}^{2\mu} + \sum_{k=2\mu+1}^{\infty}\right) \theta_{k\mu} \Delta\left(\frac{1}{\lambda_k}\right)$$

$$= O\left(\sum_{k=\mu}^{2\mu} \frac{\theta_{k\mu}}{k\lambda_k}\right) + \theta_{2\mu,\mu} \sum_{k=2\mu+1}^{\infty} \Delta\left(\frac{1}{\lambda_k}\right)$$

$$= O\left(\frac{1}{\lambda_\mu}\right) \frac{1}{\mu} \sum_{k=\mu}^{2\mu} \theta_{k\mu} + \theta_{2\mu,\mu}. \frac{1}{\lambda_\mu} = O\left(\frac{1}{\lambda_\mu}\right)$$

by hypotheses (2.27) and (2.28).

This completes the proof.

3. MODIFIED NÖRLUND MEANS

We shall now study conditions on λ and p such that $| N', p, \lambda | \Leftrightarrow | N, p |$.

Recall that Σa_n is said to be summable $| N', p, \lambda |$ and written as $\Sigma a_n \in | N', p, \lambda |$, if

$$\sum \frac{| \tau_n |}{\lambda_n} < \infty$$

where

$$\tau_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \lambda_k a_k$$

We may define (N', p, λ) method to be absolutely conservative if

$$\Sigma \mid a_n \mid < \infty \Rightarrow \Sigma a_n \in \mid N', p, \lambda \mid .$$

It is easily verified that the modified Nörlund means (N', p, λ) is absolutely conservative if and only if

$$J_k = \lambda_k \sum_{n=k}^{\infty} \frac{\mid p_{n-k} \mid}{\lambda_n \mid P_n \mid} \leqslant K \qquad \ldots(3.1)$$

for $k = 0, 1, 2, \ldots.$

The following Lemma gives simpler conditions in which (3.1) holds.

Lemma 4—Let (λ_n) be positive and non-decreasing and such that

$$\sum_{n=k}^{\infty} \frac{1}{n\lambda_n} = O\left(\frac{1}{\lambda_k}\right) \qquad \ldots(3.2)$$

and let (p_n) satisfy the conditions :

(i) $P_n^* = \sum_{k=0}^{n} \mid p_k \mid = O(\mid P_n \mid)$

(ii) $(n+1)\, p_n = O(\mid P_n \mid)$.

Then (N', p, λ) is absolutely conservative.

PROOF : From (3.1)

$$J_k = \left(\sum_{n=k}^{2k} + \sum_{n=2k+1}^{\infty}\right) \frac{\lambda_k \mid p_{n-k} \mid}{\lambda_n \mid P_n \mid}$$

$$\leqslant \frac{K}{\mid P_k \mid} \sum_{n=k}^{2k} \mid p_{n-k} \mid + K.\, \lambda_k \sum_{n=2k+1}^{\infty} \frac{\mid p_{n-k} \mid}{\lambda_n \mid P_{n-k} \mid}$$

$$\leqslant K + K.\, \lambda_k \sum_{n=2k+1}^{\infty} \frac{1}{(n-k)\, \lambda_{n-k}} \leqslant K$$

by hypotheses.

This completes the proof.

Remark : The hypothesis (3.2) is automatically satisfied if (λ_n) is positive and there is a constant $\alpha > 0$ such that for $n \geqslant 1$

$$\left(\frac{\lambda_n}{n^\alpha}\right) \text{ is non-decreasing.}$$

For

$$\sum_{n=k}^{\infty} \frac{1}{n\lambda_n} = \sum_{n=k}^{\infty} \frac{n^\alpha}{n^{\alpha+1}\lambda_n} \leqslant \frac{n^k}{\lambda_k} \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+1}} = O\left(\frac{1}{\lambda_k}\right).$$

We now obtain the following fundamental lemma.

Lemma 5—(a) In order that

$$| N', p, \lambda | \Rightarrow | N, p |$$

it is necessary and sufficient that, for $k \geqslant 0$,

$$J_k = \lambda_k \, | P_k | \sum_{n=k}^{\infty} \left| \sum_{\nu=k}^{n} \frac{1}{\lambda_\nu} \Omega(n, \nu) \, c_{\nu-k} \right| \leqslant K$$

where $\Omega(n, \nu)$ and (c_n) are defined by (1.20) and (1.21) respectively.

(b) In order that

$$| N, p | \Rightarrow | N', p, \lambda |$$

it is necessary and sufficient that

$$I_k = \sum_{n=k}^{\infty} \frac{1}{\lambda_n \, | P_n |} \left| \sum_{\nu=k}^{n} \lambda_\nu \, p_{n-\nu} \left(\sum_{\mu=\nu}^{k-1} p_\mu \, c_{\nu-\mu} - P_{k-1} \, c_{\nu-k} \right) \right|$$
$$\leqslant K, \; k = 1, 2, 3, \ldots.$$

PROOF : When A is the (N, p) matrix, we have

$$a^{-1}_{nk} = \begin{cases} P_k \, c_{n-k} & (k \leqslant n) \\ 0 & (k > n); \end{cases}$$

$$\hat{a}_{nk} = \Omega(n, k);$$

$$b_{nk} = \frac{\lambda_k \, p_{n-k}}{\lambda_n \, P_n};$$

$$\hat{a}^{-1}_{nk} = \sum_{r=k}^{n} P_r \, (c_{n-r} - c_{n-r-1})$$

$$= - \sum_{r=0}^{k-1} P_r \, (c_{n-r} - c_{n-r-1})$$

(*equation continued on p.* 363)

$$= P_{k-1}\, c_{n-k} - \sum_{r=0}^{k-1} p_r\, c_{n-r};$$

$$b_{nk}^{-1} = \frac{\lambda_k\, P_k\, c_{n-k}}{\alpha_n}.$$

The lemma now follows from Lemma 1.

It may be noted that the condition $p_n \in \mathcal{M}$ covers the Cesàro case $p_n = A_n^{\alpha-1}$ $0 < \alpha \leqslant 1$ whereas the condition $\nabla^h p_n \in \mathcal{M}$ covers the Cesàro cases : $h < \alpha \leqslant h + 1$, $h = 0, 1, 2, \ldots$.

Now we state our main theorems.

Theorem 4—Let $\lambda_n > 0$ and increasing to ∞. Let h be a non-negative integer and let

$$\nabla^h p_n \in \mathcal{M} \qquad \ldots(3.3)$$

and

$$n^{h+1}\, \lambda_n\, \Delta^{h+1} \left(\frac{1}{\lambda_n} \right) = O\,(1). \qquad \ldots(3.4)$$

In the case $h = 0$, we further assume that (3.2) holds, i. e.

$$\sum_{n=k}^{\infty} \frac{1}{n\, \lambda_n} = O\left(\frac{1}{\lambda_k} \right).$$

Then $| N', p, \lambda | \Rightarrow | N, p |$.

Theorem 5--Let (λ_n) satisfy (3.2), hold. Let h be a non-negative integer and let $\nabla^h p_n \in \mathcal{M}$. Also let

$$n^{h+1}\, \Delta^{h+1}\, \lambda_n = O\, (\lambda_n). \qquad \ldots(3.5)$$

In the case $h = 0$, we further assume that either

$$\sum_{n=k}^{\infty} \frac{1}{nP_n} = O\left(\frac{1}{P_k} \right) \qquad \ldots(3.6)$$

or

$$\left(\frac{\lambda_n}{n} \right) \text{ is non-decreasing.} \qquad \ldots(3.7)$$

Then

$$| N, p | \Rightarrow | N', p, \lambda |.$$

Remarks : (i) It may be seen that a positive sequence λ condition (3.2) is automatically satisfied whenever (3.7) holds.

(ii) The hypotheis (3.6) is satisfied in the case $P_n = A_n^{\alpha}$, $\alpha > 0$ and is not satisfied when $P_n = \sum_{\nu=0}^{n} \frac{1}{\nu + 1}$.

Theorem 6—Let (λ_n) satisfy (3.2) and let $\lambda_n \to \infty$, $n \to \infty$. Let h be a non-negative integer such that $\nabla^h p_n \in \mathfrak{M}'$ and (3.4). hold. In the case $h = 0$, we further assume that either (3.6) or (3.7) holds. Then $| N, p | \Leftrightarrow | N', p, \lambda |$.

4. Lemmas

We need the following lemmas for the proof of Theorem 4.

Lemma 6—Let h be a positive integer, $\nabla^h p_n = q_n$ and $Q_n = q_0 + q_1 + \ldots + q_n$. If $q_n > 0$ and

Q_n/q_n is non-decreasing ...(4.1)

then

P_n/p_n is non-decreasing. ...(4.2)

Proof : It is enough to prove the lemma for $h = 1$, the general result will then follow from the repeated applications of this case. Changing the notation in an obvious way and writing

$$P_n^{(1)} = P_0 + P_1 + \ldots + P_n$$

it is enough to prove that if $p_n > 0$ and P_n/p_n is non-decreasing, then $P_n^{(1)}/P_n$ is non-decreasing. The result to be proved can be put in the form that

$$P_{n+1}^{(1)} \ P_n - P_{n+1} \ P_n^{(1)} \geqslant 0. \quad \ldots(4.3)$$

We prove (4.3) by induction on n. It is trivial that (4.3) holds when $n = 0$. Let now $n \geqslant 1$ and suppose that (4.3) holds with n replaced by $n - 1$.

We have by hypothesis

$$\frac{P_n}{p_n} \leqslant \frac{P_{n+1}}{p_{n+1}}$$

so that

$$P_n \, (P_{n+1} - P_n) \leqslant P_{n+1} \, (P_n - P_{n-1});$$

that is to say

$$P_{n+1} P_{n-1} \leqslant P_n^2 . \qquad \ldots(4.4)$$

Thus by (4.4)

$$\begin{aligned} P_{n+1}^{(1)} P_n - P_{n+1} P_n^{(1)} &= (P_n^{(1)} + p_{n+1}) P_n - P_{n+1} P_n^{(1)} \\ &= P_n^{(1)} P_n + p_{n+1} (P_n - P_n^{(1)}) \\ &= P_n^{(1)} P_n - p_{n+1} P_{n-1}^{(1)} \\ &\geqslant P_n^{(1)} P_n - \frac{P_n^2}{P_{n-1}} P_{n-1}^{(1)} \\ &= \frac{P_n}{P_{n-1}} \left(P_n^{(1)} P_{n-1} - P_n P_{n-1}^{(1)} \right) \geqslant 0 \end{aligned}$$

by the induction hypothesis.

This completes the proof of the lemma.

Lemma 7—Let h be a non-negative integer and let $\Delta^h p_n \in \mathcal{M}$. Then

(i) $c_0^{(h)} = c_0 > 0,\ c_n^{(h)} \leqslant 0\ (n \geqslant 1)$;

(ii) $c_n^{(h+1)} - c_{n+1}^{(h+1)} \geqslant 0$ for $n \geqslant 0$;

(iii) $\sum_{n=0}^{\infty} c_n^{(h)} x^n$ is absolutely convergent for $|x| \leqslant 1$;

(iv) $\sum_{k=n+1}^{\infty} | c_k^{(h)} | \leqslant c_n^{(h+1)}$;

(v) $(\nabla^{h-1} p_n)\, c_n^{(h+1)} \leqslant 1$;

(vi) $(\nabla^{h-1} p_n)\, c_n^{(h+2)} \leqslant 2n + 1$;

(vii) $0 \leqslant \sum_{\mu=k}^{r} (\Delta_\mu^h p_{n-\mu})\, c_{\mu-k}^{(h)} \leqslant (\Delta_k^h p_{n-k})\, c_{\nu-k}^{(h+1)}$, $0 \leqslant k \leqslant r \leqslant n$.

PROOF : This follows from Kaluza's theorem (see Hardy[5], Theorem 22), the identity (obtained from (1.21)) :

$$\left(\sum_{n=0}^{\infty} c_n^{(h)} x^n\right)\left(\sum_{n=0}^{\infty} (\nabla^h p_n) x^n\right) = 1 \qquad \ldots(4.5)$$

and from the Lemmas 3, 4, 6 of Das[4] (see also Das[3], Lemmas 1 and 2).

Lemma 8—Let $\nabla^h p_n \in \mathfrak{M}$. Let $0 \leqslant s < h$ and $0 \leqslant r \leqslant h$. Then

(i) $\nabla^s p_n > 0$, non-decreasing and

$\nabla^s p_n \to \infty$ as $n \to \infty$;

(i) $(n + h - r + 1)\, \nabla^r p_n \leqslant (h - r + 1)\, \nabla^{r-1} p_n$;

(iii) $$\nabla^r p_n \leqslant \frac{h!\,(n + h - r)!}{(n + h)!\,(h - r)!}\, p_n$$

$$\leqslant \frac{(h + 1)!\,(n + h - r)!}{(h - r)!\,(n + h + 1)!}\, P_n = O\left(\frac{P_n}{n^{r+1}}\right);$$

(iv) $$P_n = \sum_{k=0}^{n} A_{n-k}^{h-1}\, (\Delta^{h-1} p_k) \leqslant (\nabla^{h-1} p_n)\, A_n^h\,;$$

and

$$P_n = \sum_{k=0}^{n} A_{n-k}^{h}\, \nabla^h p_k \geqslant \nabla^h p_n.\ A_n^{h+1}\,;$$

(v) $$\frac{\nabla^h p_n}{\nabla^{h-1} p_n} = O\left(\frac{p_n}{P_n}\right);$$

(vi) The method (N, p) is regular and absolutely regular.

PROOF: Since

$$\nabla^r p_n = \sum_{\nu=0}^{n} A_{n-\nu}^{h-r-1}\, \nabla^h p_\nu \qquad \ldots(4.6)$$

and since $\nabla^h p_n > 0$, $A_\nu^{h-r-1} > 0$, the result (i) follows. To prove (ii), we proceed as in Das[3] (Lemma 3). It follows from (4.6) that

$$(h - r + 1)\, \nabla^{r-1} p_n - (n + h - r + 1)\, \nabla^r p_n$$

$$= \sum_{\nu=0}^{n} \{(h - r + 1)\, A_{n-\nu}^{h-r} - (n + h - r + 1)\, A_{n-\nu}^{h-r-1}\}\, \nabla^h p_\nu. \qquad \ldots(4.7)$$

But

$$\sum_{\nu=0}^{n} \{(h - r + 1) A_{n-\nu}^{h-r} - (n + h - r + 1) A_{n-\nu}^{h-r-1}\}$$

$$= (h - r + 1) A_n^{h-r+1} - (n + h - r + 1) A_n^{h-r} = 0 \qquad \dots(4.8)$$

as

$$\frac{A_n^{h-r+1}}{A_n^{h-r}} = \frac{n + h - r + 1}{h - r + 1}$$

which is an increasing function of n. Hence it follows that there exists $\nu \leqslant \nu_0 (n)$ such that the expression bracketed in (4.7) is non-negative for $\nu \leqslant \nu_0$ and negative for $\nu > \nu_0$. Hence the right of (4.7) is greater than or equal to

$$\sum_{\nu=0}^{n} \{(h - r + 1) A_{n-\nu}^{h-r} - (n + h - r + 1) A_{n-\nu}^{h-r-1}\} \nabla^h p_{\nu_0}$$

which is 0 by another application of (4.8). Now (ii) follows. (iii) follows by repeated application of (ii). (iv) is obvious. (vi) follows from (iii) and (iv).

Since $p_n > 0$, the necessary and sufficient condition for (N, p) to be regular is that

$$p_n = o(P_n) \qquad \dots(4.9)$$

and this holds because of (iii). To prove the absolute regularity, in view of (4.9) it is enough to show that, uniformly in $\nu \geqslant 0$,

$$\sum_{n=\nu}^{\infty} | \Omega(n, \nu) | = O(1). \qquad \dots(4.10)$$

Since

$$\Omega(n, \nu) = \frac{P_{n-\nu}}{P_n} - \frac{P_{n-\nu-1}}{P_{n-1}}$$

$$= \frac{P_n p_{n-\nu} - P_{n-\nu} p_n}{P_n P_{n-1}}$$

$$= \frac{p_n p_{n-\nu} \left(\frac{P_n}{p_n} - \frac{P_{n-\nu}}{p_{n-\nu}} \right)}{P_n P_{n-1}}$$

it follows that, as $p_n \geqslant 0$.

$$\Omega(n, \nu) \geqslant 0$$

if and only if P_n/p_n is non-decreasing. But since $\nabla^h p_n = q_n$ is non-increasing, it follows that Q_n/q_n is non-decreasing. Hence, by Lemma 6, it follows that P_n/p_n is non-decreasing. Hence $\Omega(n, \nu) \geqslant 0$. So

$$\sum_{n=\nu}^{\infty} |\Omega(n, \nu)| = \sum_{n=\nu}^{\infty} \Omega(n, \nu) = \lim_{m\to\infty} \frac{P_{m-\nu}}{P_m} = 1$$

by use of (4.9).

This completes the proof.

Lemma 9—(*a*) Let h be a non-negative integer such that $\nabla^h p_n \in \mathfrak{M}$. If $h = 0$ we further assume that (3.6) holds. Then

$$P_k \sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}} = O(1).$$

(b) Let $\nabla^h p_n \in \mathfrak{M}$ and let (λ_n) satisfy (3.2). Then

$$\lambda_k\ P_k \sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}\lambda_\nu} = O(1).$$

PROOF : Let $h \geqslant 1$. Since $c_n^{(h+1)}$ is non-increasing we obtain

$$\sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}} = \left(\sum_{\nu=k}^{2k} + \sum_{\nu=2k+1}^{\infty}\right) \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1}}$$

$$\leqslant \frac{c_k^{(h+2)}}{(k+1)^{h+1}} + c_k^{(h+1)} \sum_{\nu=2k+1}^{\infty} \Bigg) \frac{1}{(\nu+1)^{h+1}}$$

$$= O\left(\frac{k+1}{\nabla^{h-1} p_k}\right) \frac{1}{(k+1)^{h+1}} + O(1)\left(\frac{1}{\nabla^{h-1} p_k}\right) \frac{1}{(k+1)^h}$$

$$= O(1/P_k)$$

by Lemma 7(v), (vi) and Lemma 8 (iv).

If $h = 0$, then we split the sigma as before and obtain

$$\sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(1)}}{(\nu+1)} \leqslant \frac{c_k^{(2)}}{k+1} + \sum_{\nu=2k+1}^{\infty} \frac{1}{(\nu+1)\, P_{\nu-k}}$$

$$= O(1/P_k)$$

by Lemma 7 (v), (vi) and hypothesis (3.6).

This proves (a). For the proof of (b), we proceed as before and obtain

$$\sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^h \lambda_\nu} = O\left(\frac{1}{\lambda_k P_k}\right) + O\left(\frac{1}{P_k}\right) \sum_{\nu=2k+1}^{\infty} \frac{1}{(\nu+1)\lambda_\nu}$$

$$= O\left(\frac{1}{\lambda_k P_k}\right)$$

by (3.2).

Lemma 10—Let h be a non-negative integer such that $\nabla^h p_n \in \mathfrak{M}$. Let $\lambda_n > 0$ non-decreasing. Then

(i) $$\sum_{n=k}^{\infty} \frac{\Delta_k^r \, p_{n-k}}{P_n} = O\left(\frac{1}{(k+1)^r}\right) \text{ for } r = 1, 2, \ldots h;$$

(ii) $$\sum_{n=k}^{\infty} \frac{\Delta_k^r \, p_{n-k}}{(n+1)P_n} = O\left(\frac{1}{(k+1)^{r+1}}\right) \text{ for } r = 0, 1, 2, \ldots h;$$

(iii) $$\sum_{n=k}^{\infty} \frac{\Delta_k^r \, p_{n-k}}{\lambda_n P_n} = O\left(\frac{1}{(k+1)^r \lambda_k}\right) \text{ for } r = 1, 2, \ldots h.$$

The result (iii) remains valid for $r = 0$ with the additional restriction contained in (3.2) [see Lemma 4].

PROOF : By Lemma 8 (i), (iii)

$$\sum_{n=k}^{\infty} \frac{\Delta_k^r \, p_{n-k}}{P_n} = \left(\sum_{n=k}^{2k} + \sum_{n=2k+1}^{\infty}\right) \frac{\Delta_k^r \, p_{n-k}}{P_n}$$

$$\leqslant \frac{1}{P_k} \sum_{n=k}^{2k} \Delta_k^r \, p_{n-k} + K \sum_{n=2k+1}^{\infty} \frac{P_{n-k}}{(n-k)^{r+1} P_n}$$

$$\leqslant \frac{\nabla^{r-1} p_k}{P_k} + K \sum_{n=2k+1}^{\infty} \frac{1}{(n-k)^{r+1}}$$

$$\leqslant \frac{K}{(k+1)^r}.$$

This complets the proof of (i). The results (ii) and (iii) are immediate corollaries of (i) for $r \geqslant 1$. When $r = 0$, the result (ii) can be proved as in Lemma 4 by use of Lemma 8 (iii).

Lemma 11—Let h be a non-negative integer and let $\nabla^h p_n \in \mathfrak{M}$. Then

$$\sum_{n=k}^{\infty} \mid \Delta_k^r \Omega(n, k) \mid = O\left(\frac{1}{(k+1)^r}\right)$$

for $r = 0, 1, 2, \ldots h$.

PROOF: The case $r = 0$ is (4.10) which has been proved earlier in the proof of Lemma 8 (vi). Consider the case $r \geqslant 1$.

Since

$$\Delta_k^r \Omega(n, k) = \frac{\Delta_k^{r-1} p_{n-k}}{P_n} - \frac{\Delta_k^{r-1} p_{n-k-1}}{P_{n-1}}$$

$$= \frac{\Delta_k^r p_{n-k}}{P_n} - \frac{p_n}{P_n P_{n-1}} \Delta_k^{r-1} p_{n-k-1}$$

we obtain :

$$\sum_{n=k}^{\infty} \mid \Delta^r \Omega(n, k) \mid \leqslant \sum_{n=k}^{\infty} \frac{\Delta_k^r p_{n-k}}{P_n} + \sum_{n=k}^{\infty} \frac{p_n}{P_n P_{n-1}} \Delta_k^{r-1} p_{n-k-1}$$

$$= O\left(\frac{1}{(k+1)^r}\right) + O(1) \sum_{n=k}^{\infty} \frac{\Delta_k^{r-1} p_{n-k-1}}{(n+1) P_n}$$

$$= O\left(\frac{1}{(k+1)^r}\right)$$

by Lemma 8 (iii) and Lemma 10 (ii).

Lemma 12—Let h be a non-negative integer, $\nabla^h p_n \in \mathfrak{M}$ and let $\lambda_n > 0$ be such that λ_n increases to ∞ and

$$n(\lambda_{n+1} - \lambda_n) = O(\lambda_n).$$

Then

$$g_k = \sum_{n=k}^{\infty} \frac{(\lambda_{n+h+1} - \lambda_{k+h+1})}{\lambda_{n+h+1}} \frac{\Delta_k^{h+1} p_{n-k}}{P_n} = O\left(\frac{1}{(k+1)^{h+1}}\right).$$

$$h_k = \sum_{n=k}^{\infty} \frac{(\lambda_{n+h+1} - \lambda_{k+h+1})}{\lambda_{n+h+1}} \frac{\Delta_k^{h} p_{n-k-1}}{n P_n} = O\left(\frac{1}{(k+1)^{h+1}}\right).$$

Proof: Write $q_n = \nabla^h p_n$. Now by Abel's transformation

$$0 \leqslant \beta(n, k) = \sum_{\mu=k+1}^{n} (\lambda_{\mu+h+1} - \lambda_{k+h+1})(q_{\mu-k-1} - q_{\mu-k})$$

$$= - \sum_{\mu=k}^{n} \Delta_\mu (\lambda_{\mu+h+1})\, q_{\mu-k} - (\lambda_{n+h+2} - \lambda_{k+h+1})\, q_{n-k}$$

$$= \sum_{\mu=k}^{n} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\, q_{\mu-k} - (\lambda_{n+h+2} - \lambda_{k+h+1})\, q_{n-k}$$

$$< \sum_{\mu=k}^{n} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\, q_{\mu-k} \qquad \ldots(4.11)$$

as $q_n > 0$ and non-increasing.

As $(R, \lambda_n, 1)$ is regular, it follows that

$$R_n = \frac{1}{\lambda_{n+h+1}} \sum_{\mu=k}^{n} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\, q_{\mu-k}$$

converges to a limit as $n \to \infty$. Now since $P_n \to \infty$ (by Lemma 8 (i)), we have from (4.11) that

$$\frac{\beta(n, k)}{\lambda_{n+h+1} P_n} \leqslant \frac{R_n}{P_n} = o(1) \qquad \ldots(4.12)$$

as $n \to \infty$, for any fixed k.

Now by Abel's transformation, (4.11) and (4.12), we have

$$g_k = \sum_{n=k+1}^{\infty} \Delta_n \left(\frac{1}{\lambda_{n+h+1} P_n} \right) \beta(n, k)$$

$$\leqslant \sum_{\mu=k}^{\infty} (\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\, q_{\mu-k} \sum_{n=\mu}^{\infty} \Delta_n \left(\frac{1}{\lambda_{n+h+1} P_n} \right)$$

$$\leqslant \sum_{\mu=k}^{\infty} \frac{(\lambda_{\mu+h+2} - \lambda_{\mu+h+1})\, q_{\mu-k}}{\lambda_{\mu+h+1} P_\mu}$$

$$\leqslant K \sum_{\mu=k}^{\infty} \frac{q_{\mu-k}}{\mu P_\mu} \leqslant \frac{K}{(k+1)^{h+1}}$$

by Lemma 10 (ii).

Next,

$$h_k \leqslant \sum_{n=k+1}^{\infty} \frac{\Delta_k^h \, p_{n-k-1}}{n \, P_n} = O\left(\frac{1}{(k+1)^{h+1}} \right)$$

by Lemma (10) (ii).

This completes the proof of Lemma 12.

Lemma 13—Let h be any non-negative integer such that $\nabla^h p_n \in \mathscr{M}$ and let $\lambda_n > 0$ non-decreasing and

$$n\,(\lambda_{n+1} - \lambda_n) = O\,(\lambda_n).$$

Then

$$S_k = \sum_{n>k} \left(\frac{1}{\lambda_{k+h+1}} - \frac{1}{\lambda_{n+h+1}} \right) | \Delta_k^{h+1} \, \Omega\,(n, k) |$$

$$= O\left(\frac{1}{(k+1)^{h+1} \, \lambda_k} \right).$$

PROOF : Since

$$\Delta_k^{h+1} \, \Omega\,(n, k) = \frac{\Delta_k^{h+1} \, p_{n-k}}{P_n} - \frac{p_n}{P_n \, P_{n-1}} \, \Delta_k^h \; p_{n-k-1}$$

and since by Lemma 8 (iii)

$$\frac{p_n}{P_{n-1}} = O\left(\frac{1}{n} \right).$$

It follows that

$$S_k \leqslant \frac{K}{\lambda_k} (g_k + h_k)$$

$$\leqslant \frac{K}{(k+1)^{h+1} \, \lambda_k}$$

by Lemma 12.

This completes the proof.

Lemma 14—Let $\lambda_n > 0$ and non-decreasing. Let h be a non-negative integer. Then

$$n^{h+1} \, \lambda_n \, \Delta^{h+1} \left(\frac{1}{\lambda_n} \right) = O\,(1)$$

implies that

$$n^{r+1}\,\lambda_n\,\Delta^{r+1}\left(\frac{1}{\lambda_n}\right)=O(1)\ \text{ for } r=0,1,2,\ldots h-1.$$

PROOF : This can be proved as in Chow[2] (Lemma 13).

5. PROOF OF THEOREM 4

By Lemma 5 (a), it is enough to prove that J_k is bounded. By effecting $h+1$ times successive Abel's transformation, we obtain :

$$\sum_{\nu=k}^{n}\frac{1}{\lambda_\nu}\,\Omega(n,\nu)\,c_{\nu\,k}$$

$$=\sum_{\nu=k}^{n}\Delta_\nu^{h+1}\left(\frac{\Omega(n,\nu)}{\lambda_\nu}\right)c_{\nu-k}^{(h+1)}$$

$$=\sum_{r=0}^{h+1}\binom{h+1}{r}\sum_{\nu=k}^{n}\left(\Delta_\nu^{h+1-r}\,\frac{1}{\lambda_{\nu+r}}\right)\Delta_\nu^{r}\,\Omega(n,\nu)\,c_{\nu-k}^{(h+1)}\,.$$

We now consider the terms $r=0$ to $r=h$ together and the term $r=h+1$ separately in the last sigma and obtain

$$J_k\leqslant J_k^{(1)}+J_k^{(2)}$$

where

$$J_k^{(1)}=\lambda_k\,P_k\sum_{n=k}^{\infty}\left|\sum_{\nu=k}^{n}\frac{1}{\lambda_{\nu\ h+1}}\left(\Delta_\nu^{h+1}\,\Omega(n,\nu)\,c_{\nu-k}^{(h+1)}\right|\right.$$

$$J_k^{(2)}=\sum_{r=0}^{h}\binom{h+1}{r}\lambda_k\,P_k\sum_{n=k}^{\infty}\left|\sum_{\nu=k}^{n}\Delta_\nu^{h+1-r}\left(\frac{1}{\lambda_{\nu+1r}}\right)\right.$$

$$\left.\times\,\Delta_\nu^{r}\,\Omega(n,\nu)\,c_{\nu-k}^{(h+1)}\right|.$$

Writting

$$\frac{1}{\lambda_{\nu+h+1}}=\frac{1}{\lambda_{\nu+h+1}}-\frac{1}{\lambda_{n+h+1}}+\frac{1}{\lambda_{n+h+1}}$$

we obtain

$$J_k^{(1)} \leqslant J_k^{(11)} + J_k^{(12)}$$

where

$$J_k^{(11)} = \lambda_k P_k \sum_{n=k}^{\infty} \Big| \sum_{\nu=k}^{n} \left(\frac{1}{\lambda_{\nu+h+1}} - \frac{1}{\lambda_{n+h+1}} \right) \left(\Delta_\nu^{h+1} \Omega(n, \nu) \right) \times c_{\nu-k}^{(h+1)} \Big|$$

$$= O(1) \lambda_k P_k \sum_{n=\nu}^{\infty} c_{\nu-k}^{(h+1)} \sum_{n=\nu}^{\infty} \left(\frac{1}{\lambda_{\nu+h+1}} - \frac{1}{\lambda_{n+h+1}} \right) \times | \Delta_\nu^{h+1} \Omega(n, \nu) |$$

$$= O(1) \lambda_k P_k \sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1} \lambda_\nu} = O(1).$$

by Lemmas 13 and 9. And

$$J_k^{(12)} = \lambda_k P_k \sum_{n=k}^{\infty} \frac{1}{\lambda_{n+h+1}} \Big| \sum_{\nu=k}^{n} \Delta_\nu^{h+1} \Omega(n, \nu) c_{\nu-k}^{(h+1)} \Big| .$$

It follows from (4.5) that

$$\sum_{\nu=k}^{n} c_{\nu-k}^{(h+1)} \Delta_\nu^{h+1} P_{n-\nu} = 1 \quad (n \geqslant k)$$

and so

$$\sum_{\nu=k}^{n} c_{\nu-k}^{(h+1)} \Delta_\nu^{h+1} \Omega(n, \nu) = \frac{1}{P_n} - \frac{1}{P_{n-1}} .$$

Hence

$$J_k^{(12)} = O(1) \lambda_k P_k \sum_{n=k}^{\infty} \frac{1}{\lambda_{n+h+1}} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right)$$

(*equation continued on p.* 375)

$$= O(1)\, P_k \sum_{n=k}^{\infty} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right)$$

$$= O(1).$$

Therefore $J_k^{(1)}$ is bounded.

Lastly

$$J_k^{(2)} = \sum_{r=0}^{h} \binom{h+1}{r} \lambda_k\, P_k \sum_{\nu=k}^{\infty} c_{\nu-k}^{(h+1)} \left| \Delta_\nu^{h+1-r} \left(\frac{1}{\lambda_{\nu+r}} \right) \right|$$

$$\times \left| \sum_{n=\nu}^{\infty} \left| \Delta_\nu^{r}\, \Omega(n, \nu) \right| \right.$$

$$= O(1)\, \lambda_k\, P_k \sum_{\nu=k}^{\infty} c_{\nu-k}^{(h+1)} \frac{1}{(\nu+1)^r} \left| \Delta_\nu^{h+1-r} \left(\frac{1}{\lambda_{\nu+r}} \right) \right|$$

$$= O(1)\, \lambda_k\, P_k \sum_{\nu=k}^{\infty} \frac{c_{\nu-k}^{(h+1)}}{(\nu+1)^{h+1} \lambda_\nu} = O(1)$$

by hypotheses and Lemmas 11, 14 and 9.

This completes the proof of Theorem 4.

6. Further Lemmas

We need the following additional lemmas for the proofs of Theorems 5 and 6.

Lemma 15—Let h be a non-negative integer and let $\nabla^h p_n \in \mathfrak{M}$.

Then

$$M_k = \sum_{n=k}^{\infty} \frac{\left(\Delta_k^h\, p_{n-k} \right) c_{n-k}^{(h+2)}}{(n+h)\, P_n} = O\left(\frac{1}{P_k} \right).$$

Proof : Since $c_n^{(h+2)}$ is non-decreasing we obtain :

$$M_k = \left(\sum_{n=k}^{2k} + \sum_{n=2k+1}^{\infty} \right) \frac{\left(\Delta_k^h\, p_{n-k} \right) c_{n-k}^{(h+2)}}{(n+h)\, P_n}$$

(*equation continued on p.* 376)

$$\leqslant \frac{c_k^{(h+2)}}{(k+h)\, P_k} \sum_{n=k}^{2k} \Delta_k^h \, p_{n-k} + K \sum_{n=2k+1}^{\infty}$$

$$\times \frac{(n-k+1)\, \Delta_k^h \, p_{n-k}}{(n+h)\, P_n \, \Delta_k^{h-1} \, p_{n-k}}$$

$$\leqslant \frac{c_k^{(h+2)} \quad \nabla^{h-1} p_k}{(k+h)\, P_k} + K \sum_{n=2k+1}^{\infty} \frac{(n-k+1)\, p_{n-k}}{(n+h)\, P_n \, P_{n-k}}$$

$$\leqslant \frac{K}{P_k} + \sum_{n=k+1}^{\infty} \left(\frac{1}{P_{n-k-1}} - \frac{1}{P_{n-k}} \right) \leqslant \frac{K}{P_k}$$

by hypotheses, Lemma 7 (vi) and Lemma 8 (v).

This completes the proof.

Lemma 16—Let h be a non-negative integer and let (λ_n) be a positive non-decreasing sequence such that

$$n^{h+1} \, \Delta^{h+1} \, \lambda_n = O\,(\lambda_n).$$

Then

$$n^{r+1} \, \Delta^{r+1} \, \lambda_n = O\,(\lambda_n)$$

for

$$r = 0, 1, 2, \ldots h-1.$$

Proof : It is enough to prove the result for $r = h - 1$, since the general result can then be obtained by repeated applications of this case. The proof is based on one verson of the 'discrete' analogue of Taylor's theorem with remainder and is given by the following formula, valid for $0 \leqslant k \leqslant n$:

$$\lambda_{n-k} = \sum_{r=0}^{h} \binom{k+r-1}{r} \Delta^r \lambda_n + \sum_{\nu=n-k}^{n-1} \binom{\nu+h-n+k}{h} \Delta^{h+1} \lambda_\nu. \qquad \ldots(6.1)$$

This is easily proved by induction on $h = 0$; and if in the second term in (6.1), we substitute

$$\Delta^{h+1} \lambda_\nu = \Delta^{h+1} \lambda_n + \sum_{\mu=\nu}^{n-1} \Delta^{h+2} \lambda_\mu$$

we obtain, after simplification, the same formula as (6.1) but with h replaced by $h+1$. This establishes (6.1).

Note that if c is a constant such that $0 < c < 1$, then for $0 \leqslant k \leqslant cn$ and $n-k \leqslant \nu \leqslant n-1$, we have $\nu \geqslant \alpha\, n$ where $\alpha > 1-c > 0$. Hence

$$\Delta^{h+1} \lambda_\nu = O\left(\frac{\lambda_\nu}{\nu^{h+1}}\right) = O\left(\frac{\lambda_\nu}{n^{h+1}}\right) = O\left(\frac{\lambda_n}{n^{h+1}}\right)$$

as (λ_ν) is non-decreasing. Since the sum of the coefficients in the second sum in (6.1) is

$$\binom{k+h}{h+1} = O\left(k^{h+1}\right) = O\left(n^{h+1}\right)$$

it follows that, uniformly in $0 \leqslant k \leqslant cn$, we have

$$I\,(n, k) = \sum_{r=0}^{h} \binom{k+r-1}{r} \Delta^r \lambda_n = O\,(\lambda_n). \qquad \ldots(6.2)$$

Now let d be a constant with $0 < d < \frac{1}{h}$, and let $k = [nd]$. It follows from (6.2) that

$$\sum_{\nu=0}^{h} (-1)^\nu \binom{h}{\nu} I\,(n, \nu k) = O\,(\lambda_n) \qquad \ldots(6.3)$$

since the coefficients are constants. Now the sum in the left of (6.3) is the hth difference of $I\,(n, \nu k)$ (regarded as a function of ν) taken for $\nu = 0$. The terms with $r < h$ in the sum defining $I\,(n, \nu k)$ are polynomials in ν of degree less than h; their hth differences are therefore 0. The terms with $r = h$ is a polynomial of degree h, the coefficient of ν^h being

$$(-1)^h \frac{k^h}{h!} \Delta^h \lambda_n$$

its hth difierence is therefore

$$k^h \Delta^h \lambda_n.$$

Hence (6.3) reduces to

$$\Delta^h \lambda_n = O\left(\frac{\lambda_n}{k^h}\right) = O\left(\frac{\lambda_n}{n^h}\right)$$

by definition of k. Hence the result.

Lemma 17—Let $\lambda_n > 0$ and non-decreasing. Then the hypotheses (3.4) and (3.5) are equivalent; i.e.

$$n^{h+1} \lambda_n \Delta^{h+1} \left(\frac{1}{\lambda_n}\right) = O\,(1)$$

if and only if

$$n^{h+1}\,\Delta^{h+1}\,\lambda_n = O\,(\lambda_n).$$

PROOF : Suppose that (3.5) holds. Then by Lemma 16

$$n^{r+1}\,\Delta^{r+1}\,\lambda_n = O\,(\lambda_n)$$

for

$$r = 0, 1, 2, \ldots h - 1.$$

Write $\lambda_n^{(a)}$ for any product of terms each of the form λ_{n+b} where b is a constant integer. The values of b for different factors of the product may not be the same. We use a similar notation for $(\Delta^r\,\lambda_n)^{(a)}$. When $a = 0$, this is taken as meaning 1.

Since

$$\Delta\left(\frac{1}{\lambda_n}\right) = -\,\frac{\Delta\,\lambda_n}{\lambda_n\,\lambda_{n+1}}$$

we can verify by induction on h that $\Delta^{h+1}\left(\frac{1}{\lambda_n}\right)$ is the sum of a finite number of terms of the form :

$$\frac{(\Delta\lambda_n)^{(a_1)}\ (\Delta^2\,\lambda_n)^{(a_2)}\ ..\ (\Delta^{h+1}\,\lambda_n)^{(a_{n+1})}}{\lambda_n^{(b)}}$$

where $a_1, a_2, \ldots,$ are non-negative integers such that

$$a_1 + 2a_2 + 3a_3 + \ldots + (h+1)\,a_{h+1} = h + 1$$

$$a_1 + a_2 + \ldots + a_{h+1} + 1 = b.$$

It follows easily that (3.4) holds.

The converse implication may be proved in a similar way using the formula just stated with λ_n replaced by $1/\lambda_n$.

7. PROOF OF THEOREM 5

By Lemma 5 (b), it is enough to prove that I_k is bounded. Now

$$I_k \leqslant I_k^{(1)} + I_k^{(2)}$$

where

$$I_k^{(1)} = \sum_{n=k}^{\infty} \frac{1}{\lambda_n\,P_n} \left| \sum_{\mu=0}^{k-1} p_\mu \sum_{\nu=k}^{n} c_{\nu-\mu}\,p_{n-\nu}\,\lambda_\nu \right|$$

$$I_k^{(2)} = P_{k-1} \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left| \sum_{\nu=k}^{n} \lambda_\nu p_{n-\nu} c_{\nu-k} \right| .$$

By Abel's transformation h times.

$$\sum_{\nu=k}^{n} \lambda_\nu p_{n-\nu} c_{\nu-\mu}$$

$$= \sum_{\nu=k}^{n} \Delta_\nu(\lambda_\nu p_{n-\nu}) c_{\nu-\mu}^{(1)} - c_{k-\mu-1}^{(1)} \lambda_k p_{n-k}$$

$$= \sum_{\nu=k}^{n} \Delta_\nu^h (\lambda_\nu p_{n-\nu}) \left(c_{\nu-\mu}^{(h)} - \sum_{r=1}^{h} c_{k-\mu-1}^{(h)} \Delta_k^{r-1} (\lambda_k p_{n-k}) \right.$$

Hence

$$I_k^{(1)} \leqslant I_k^{(11)} + I_k^{(12)}$$

where

$$I_k^{(11)} = \sum_{r=0}^{h} \binom{h}{r} \sum_{\mu=0}^{k-1} p_\mu \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left| \sum_{\nu=k}^{n} (\Delta^{h-r} \lambda_{\nu+r}) (\Delta_\nu^r p_{n-\nu}) c_{\nu-\mu}^{(h)} \right|$$

$$< \sum_{r=0}^{h} \binom{h}{r} \sum_{\mu=0}^{k-1} p_\mu \sum_{\nu=k}^{\infty} (\Delta^{h-r} \lambda_{\nu+r}) \left| c_{\nu-\mu}^{(h)} \right| \sum_{n=\nu}^{\infty} \left| \frac{\Delta_\nu^r p_{n-\nu}}{\lambda_n P_n} \right|$$

$$= O(1) \sum_{r=0}^{h} \binom{h}{r} \sum_{\mu=0}^{h-1} p_\mu \sum_{\nu=k}^{\infty} \frac{\left| c_{\nu-\mu}^{(h)} \right|}{\nu^r} \frac{\left| \Delta_\nu^{h-r} \lambda_{\nu+r} \right|}{\lambda_\nu}$$

$$= O(1) \sum_{\mu=0}^{k-1} p_\mu \sum_{\nu=k}^{\infty} \frac{\left| c_{\nu-\mu}^{(h)} \right|}{\nu^r} \cdot \frac{1}{\nu^{h-r}}$$

$$= O(1) \frac{1}{k^h} \sum_{\mu=0}^{k-1} p_\mu \sum_{\nu=k}^{\infty} \left| c_{\tau-\mu}^{(h)} \right|$$

(*equation continued on p.* 380)

$$= O(1)\ \frac{1}{k^h} \sum_{\mu=0}^{k-1} p_\mu\, c_{k-\mu-1}^{(h+1)}$$

$$= O(1).\ \frac{1}{k^h}\, k^h = O(1)$$

by Lemma 8 (iii), hypothesis (3.5), Lemma 7 (iv) and by the fact that

$$\sum_{k=0}^{n} p_{n-k}\, c_k^{(h+1)} = A_n^h$$

which, by virtue of (1.21), follows from the identity :

$$(\Sigma\, p_n\, x^n)\,(\Sigma\, c_n^{(h+1)}\, x^n) = \Sigma\, A_n^h\, x^n. \qquad \ldots(7.1)$$

Before we consider $I_k^{(12)}$, note that by hypothesis (3.5) and Lemma 16, we have

$$n\,(\lambda_n - \lambda_{n+1}) = O(\lambda_n)$$

and this implies that

$$\lambda_n \sim \lambda_{n+1}.$$

Hence

$$\frac{\lambda_{n+\theta}}{\lambda_n} \to 1 \qquad \ldots(7.2)$$

as $n \to \infty$, for fixed θ.

Now

$$I_k^{(12)} = \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left| \sum_{\mu=0}^{k-1} p_\mu \sum_{r=1}^{h} c_{k-\mu-1}^{(r)}\ \Delta_k^{r-1}\,(\lambda_k\, p_{n-k}) \right|$$

$$= \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left| \sum_{\mu=0}^{k-1} p_\mu \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} c_{k-\mu-1}^{(r)} \right.$$

$$\left. \times \left(\Delta_k^{r-1-\theta}\,(\lambda_{k+\theta}) \right) \Delta_\mu^{\theta}\, p_{n-\mu} \right| \ -$$

$$\leqslant \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \left| \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left(\Delta_k^{r-1-\theta}\, \lambda_{k+\theta}\right) \right.$$

(*equation continued on p.* 381)

$$\times\ (\Delta_k^{\theta}\ p_{n-k}) \sum_{\mu=0}^{k} p_\mu\ c_{k-\mu-1}^{(r)}\ |$$

$$= O(1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \mid \Delta_k^{r-1-\theta}\ \lambda_{k+\theta} \mid$$

$$\times\ (\Delta_k^{\theta}\ p_{n-k})\, k^{r-1}$$

$$= O\ (1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \mid \Delta_k^{r-1-\theta}\ \lambda_{k+\theta} \mid k^{r-1} \sum_{n=k}^{\infty}$$

$$\times\ \frac{\Delta_k^{\theta}\ p_{n-k}}{\lambda_n P_n}$$

$$= O(1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \frac{k^{r-1}}{k^{\theta}} \mid \Delta_k^{r-1-\theta}\ \lambda_{k+\theta} \mid \frac{1}{\lambda_k}$$

$$= O(1) \sum_{r=1}^{h} \sum_{\theta=0}^{r-1} \binom{r-1}{\theta} \frac{\lambda_{k+\theta}}{\lambda_k} = O(1).$$

by Lemma 10 (ii), (7.2), the hypotheses, and the fact that for $r \geqslant 0$,

$$\sum_{k=0}^{n} p_k\ c_{n-k}^{(r)} = A_n^{r-1}.$$

(This identity follows from an identity similar to (7.1)).

Thus

$$I_k^{(1)} \leqslant I_k^{(11)} + I_k^{(12)} = O(1).$$

Now we consider $I_k^{(2)}$. By making $h+1$ times Abel's transformation we have

$$\sum_{\nu=k}^{n} \lambda_\nu\, p_{n-\nu}\ c_{\nu-k}$$

$$\sum_{r=0}^{h} \left(\begin{matrix} h \\ r \end{matrix} = \sum_{\nu=k}^{n} (\Delta_\nu^{h-r+1}\ \lambda_{\nu+r})\ (\Delta_\nu^{r}\ p_{n-\nu})\ c_{\nu-k}^{(h+1)} \right).$$

Hence

$$I_k^{(2)} = \sum_{r=0}^{h} \binom{h}{r} P_k \sum_{n=k}^{\infty} \frac{1}{\lambda_n P_n} \left| \sum_{v=k}^{n} (\Delta_v^{h-r+1} \lambda_{v+r}) \times (\Delta_v^r \, p_{n-v}) \, c_{v-k}^{(h+1)} \right|$$

$$\leqslant \sum_{r=0}^{h} \binom{h}{r} P_k \sum_{v=k}^{\infty} \left| (\Delta_v^{h-r+1} \lambda_{v+r} \right| c_{v-k}^{(h+1)} \sum_{n=v}^{\infty} \frac{\Delta_v^r \, p_{n-v}}{\lambda_n P_n}$$

$$= O(1) \sum_{r=0}^{h} \binom{h}{r} P_k \sum_{v=k}^{\infty} \frac{\left| \Delta_v^{h-r+1} \lambda_{v+r} \right| c_{v-k}^{(h+1)}}{v^r \lambda_v}$$

$$= O(1) \sum_{r=0}^{h} \binom{h}{r} P_k \sum_{v=k}^{\infty} \left| \frac{c_{v-k}^{(h+1)}}{v^{h+1}} \right. = O(1)$$

by Lemma 10 (iii), Lemma 9 (a), Lemma 16 and hypotheses.

We note that the result of Lemma 9 (a) is valid with the assumption (3.6) in the case $h = 0$, and this is not needed in the case $h > 0$.

Now we consider $I_k^{(2)}$ in the case $h = 0$, under hypothesis (3.7) in place of (3.6).

Now

$$\theta\,(n, k) = \sum_{v=k}^{n} \lambda_v \, p_{n-v} \, c_{v-k}$$

$$= \sum_{v=k}^{n} \Delta \lambda_v \sum_{\mu=k}^{v} p_{n-\mu} \, c_{\mu-k} + \lambda_{n+1} \, \delta_{n,k}$$

where

$$\delta_{nk} = \sum_{\mu=k}^{n} p_{n-\mu} \, c_{\mu-k} = \begin{cases} 1 \ (n = k) \\ O \ (n \neq k). \end{cases}$$

Now by hypothesis (3.5) and (3.7) (in the case $h = 0$) we obtain,

$$\Delta \lambda_v = O\left(\frac{\lambda_v}{v} \right) = O\left(\frac{\lambda_n}{n} \right).$$

Hence by Lemma 7 (vii)

$$\theta(n, k) = O(1) \ p_{n-k} \sum_{v=k}^{n} |\Delta \lambda_v| \, c_{v-k}^{(1)} + \lambda_{n+1} \, \delta_{nk}$$

$$= O(1) \frac{\lambda_n}{n} \ p_{n-k} \ c_{n-k}^{(2)} + \lambda_{n+1} \, \delta_{nk}.$$

Hence

$$I_k^{(2)} = O(1) P_k \sum_{n=k}^{\infty} \frac{p_{n-k} \, c_{n-k}^{(2)}}{n \, P_n} + \frac{P_k \, . \, \lambda_{k+1}}{\lambda_k \, . \, P_k}$$

$$= O(1)$$

by Lemma 15.

8. Proof of Theorem 6

This follows by combining Theorem 4 and Theorem 5, taking note of Lemma 17.

9. Corollaries

We know that Cesàro mean and the modified Cesàro mean with weight n are equal. In the following corollaries we examine special sequences λ for which Nörlund and modified Nörlund means are equivalent.

From Theorem 4, we obtain the following.

Corollary 1—Let $\nabla^h p_n \in \mathfrak{M}$ and let

$$\lambda_n = n^{\beta} (\log n)^{\delta}.$$

Then

$$| N', p, \lambda | \Rightarrow | N, p |$$

in the following cases :

(i) $h > 0$ and either $\beta > 0$, δ real or $\beta = 0$, $\delta > 0$.

(ii) $h = 0$, $\beta > 0$, δ real

In particular

$$| C', \alpha, \lambda | \Rightarrow | C, \alpha |$$

in the following cases :

(i) $\alpha > 1$ and either $\beta > 0$, δ real or $\beta = 0$, $\delta > 0$

(ii) $0 < \alpha \leqslant 1$, $\beta > 0$, δ real.

Similarly from Theorem 5, we obtain the following.

Corollary 2—Let $\nabla^h p_n \in \mathscr{M}$ and let

$$\lambda_n = n^{\beta} (\text{long } n)^{\delta}.$$

Then

$$| N, p | \Rightarrow | N', p, \lambda |$$

in the following cases :

(i) $h > 0$, $\beta > 0$, δ real

(ii) $h = 0$ and either $\beta \geqslant 1$, δ real or (3.6) holds. In particular

$$| C, \alpha | \Rightarrow | C', \alpha, \lambda |$$

if $\beta > 0$, δ real. And

$$\left| N, \frac{1}{n+1} \right| \Rightarrow \left| N', \frac{1}{n+1}, \lambda \right|$$

if $\beta \geqslant 1$, δ real.

Combining Corollaries 1 and 2, we obtain the following.

Corollary 3—Let $\nabla^h p_n \in \mathscr{M}$ and let

$$\lambda_n = n^{\beta} (\log n)^{\delta}.$$

Then

$$| N', p, \lambda | \sim | N, p |$$

in the following cases :

(i) $h > 0$, $\beta > 0$, δ real

(ii) $h = 0$ and either $\beta > 1$, δ real or (3.6) holds. In particular

$$| C', \alpha, \lambda | \sim | C, \alpha |$$

if $\beta > 0$, δ real and

$$\left| N, \frac{1}{n+1} \right| \sim \left| N', \frac{1}{n+1}, \lambda \right|$$

If $\beta \geqslant 1$, δ real.

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