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SCATTERING OF A COMPRESSIONAL WAVE AT THE CORNER OF A QUARTER SPACE

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The problem of scattering of a compressional wave at the corner of an elastic quarter space ($x \geqslant 0$, $z \geqslant 0$) has been discussed in the present paper. One face of the quarter space is free and other is assumed to be rigid not permitting the displacements across it. The technique is due to Wiener and Hopf. The scattered field possesses the character of transverse cylindrical waves. The numerical computation for the amplitude of the scattered field exhibits a sharp fall versus the small values of the wave number.

INTRODUCTION

A problem of special interest in seismology is the problem of scattering of an elastic wave at a corner of an elastic medium as it leads to a phenomena of a surface wave entering into another medium after travelling through a medium. Several authors[4, 6-11] have studied the problem of Rayleigh wave propagation in elastic wedges both theoretically and experimentally. Recently, Momoi[7] has discussed the problem of scattering of a Rayleigh wave in an elastic quarter space. He has extended his study to Rayleigh wave scattering due to a rectangular mountain[8]. The three-dimensional problem of scattering of waves in an elastic quarter space has been considered recently by Gautesen[3]. Both Momoi[7,8] and Gautesen[3] have studied the problem of scattering of surface waves in an elastic quarter space using rigorous numerical computations to get approximate results. Deshwal and Mann[1] have attempted the problem of scattering of Rayleigh waves at the corner of an elastic quarter space to obtain exact results using the technique of Wiener and Hopf.

In the present paper, it is being contemplated to study the problem of scattering of a compressional wave at the corner of a quarter space. Most of the mountains are deep-rooted with their bases in the solid mantle of the earth. They are assumed to be rigid forming the rigid boundary of the present problem. They peep out of the earth and scatter the seismic waves. Physically, mountains within the earth form the rigid boundary and the surface of the earth is the free surface of the quarter space. The Fourier transformation and the function theoretic technique due to Wiener and Hopf is the method of solution.

Basic Equations

The problem is two-dimensional. The waves propagate in the zx-plane. The x-axis is in the free surface and the z-axis being along the rigid boundary with origin at the corner of the quarter space $x \geqslant 0, z > 0$. The medium is homogeneous, isotropic and slightly dissipative. A time-harmonic two-dimensional compressional wave is incident at the corner (Fig. 1) and gives rise to the reflected and scattered waves.

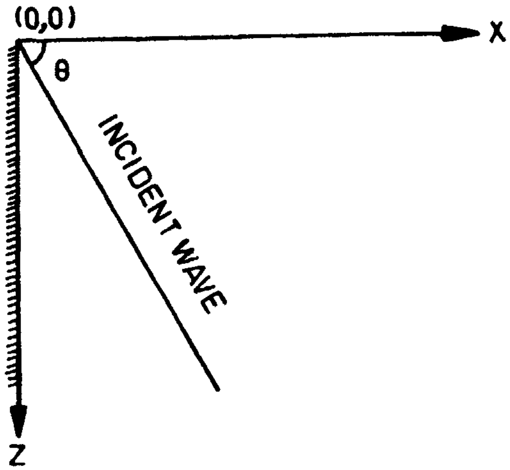

Fig. 1. A solid quarter space with a rigid boundary.

The incident wave is

$$\phi_i (x, z) = \exp[-ik(x \cos\theta + z \sin\theta)]. \qquad \ldots(1)$$

The total potentials in the medium are

$$\phi_t (x, z) = \phi_i (x, z) + \phi (x, z) \qquad \ldots(2)$$

$$\psi_t (x, z) = \psi (x, z). \qquad \ldots(3)$$

The wave equations are

$$(\nabla^2 + k^2)\phi = 0 = (\nabla^2 + k'^2)\psi. \qquad \ldots(4)$$

$k = k_1 + ik_2$, $k' = k'_1 + ik'_2$ are complex wave numbers with positive imaginary parts. The displacements (u, w) are given by

$$u = \frac{\partial\phi_t}{\partial x} - \frac{\partial\psi_t}{\partial z}, \; w = \frac{\partial\phi_t}{\partial z} + \frac{\partial\psi_t}{\partial x}. \qquad \ldots(5)$$

Boundary Conditions

The conditions on the boundaries and at distant points of the quarter space are

$$\text{(i) } u = 0 = w, \; x = 0, \; z \geqslant 0 \qquad \text{...(6)}$$

$$\text{(ii) } p_{zz} = 0 = p_{zx}, \; z = 0, \; x \geqslant 0 \qquad \text{...(7)}$$

where p_{zz}, p_{zx} are the normal and the shear stresses. The conditions (6) imply that ϕ_t and ψ_t satisfy the Cauchy-Riemann equations and are harmonic functions. Equations (4) will result in

$$\phi_t (x, z) = 0 = \psi_t (x, z), \; x = 0, \; z \geqslant 0. \qquad \text{...(8)}$$

The half-range Fourier transform

$$\bar{\phi}_+ (p, z) = \int_0^\infty \phi (x, z) \, e^{ipx} \, dx, \; p = \alpha + i\beta \qquad \text{...(9)}$$

is analytic along with its derivatives in the region $\beta > -d$ of the complex p-plane if for given z

$$| \phi (x, z) | \sim M \exp (-d \, | x |), \; M, d > 0. \qquad \text{...(10)}$$

The Fourier transform of $\psi (x, z)$ and its derivatives with respect to z are analytic in the same region.

Discussion of the Problem

Let us take the Fourier transform of the first of eqns. (4) to obtain

$$\left(\frac{d^2}{dz^2} - Y^2 \right) \bar{\phi}_+ (p, z) = \left(\frac{\partial \phi}{\partial x} \right)_0 - ip \, (\phi)_0 \qquad \text{...(11)}$$

where $Y = \pm (p^2 - k^2)^{1/2}$. The sign before the radical is such that the real part of Y is always positive for all p. The subscript 0 denotes the value at $x = 0$. The first of the conditions (8) simplifies to

$$(\phi)_0 = - (\phi_t)_0 = - \exp (-ik \, z \sin \theta). \qquad \text{...(12)}$$

A complete solution of (11), which holds when $z \to \infty$, is

$$\bar{\phi}_+ (p, z) - \bar{\phi}_+ (-p, z) = A (p) \, e^{-Yz} - \frac{2ip \exp (-ik \, z \sin \theta)}{p^2 - k^2 \cos^2 \theta} \qquad \text{...(13)}$$

Putting $z = 0$ in (13) and in its derivative with respect to z and eliminating $A (p)$ between the resulting equations to obtain

$$\bar{\phi}_+ (p) - \bar{\phi}_+ (-p) = - \left[\bar{\phi}'_+ (p) - \bar{\phi}'_+ (-p) + \frac{2pk \sin \theta}{p^2 - k^2 \cos^2 \theta} \right] - \frac{2ip}{p^2 - k^2 \cos^2 \theta} . \qquad \text{...(14)}$$

The notations $\bar{\phi}_+(p)$, $\bar{\phi}'_+(p)$ are used for $\bar{\phi}_+(p, 0)$, $\bar{\phi}'_+(p, 0)$ etc. Equation (14) is a Wiener–Hopf type functional equation to be solved for two unknown functions $\bar{\phi}_+(p)$ and $\bar{\phi}_+(-p)$.

Solution of the Wiener-Hopf Equation

Equation (14) can be written as

$$\bar{\phi}_+(p) + \frac{\bar{\phi}'_+(p)}{Y} + \frac{iY + k\sin\theta}{Y(p+k\cos\theta)} = \bar{\phi}_+(-p) + \frac{\bar{\phi}'_+(-p)}{Y} - \frac{iY + k\sin\theta}{Y(p-k\cos\theta)} \qquad \ldots(15)$$

The left-hand member is analytic in the region $\beta > -k_2\cos\theta$ and the right hand member in $\beta < k_2\cos\theta$. By analytic continuation, they represent an entire function. The points $p = \pm k$ are excluded by the branch cuts. Each member tends to zero as $|p| \to \infty$. By an extension of Liouville theorem each member is identically zero. Hence

$$\bar{\phi}_+(p) = -\frac{\bar{\phi}'_+(p)}{Y} - \frac{iY + k\sin\theta}{Y(p + k\cos\theta)}, p \neq \pm k\cos\theta. \qquad \ldots(16)$$

Similarly

$$\psi_+(p) = -\psi'_+(p)/\delta,\ \delta = \pm(p^2 - k'^2)^{1/2} \qquad \ldots(17)$$

the choice of signs for δ is same as for Y.

Taking the Fourier transforms of the conditions (7) and using (16) and (17), it is obtained that

$$-2ip\,\bar{\phi}'_+(p) + \frac{(2p^2 - k'^2)\,\psi'_+(p)}{\delta} = \frac{2ipk\sin\theta}{p-k\cos\theta} \qquad \ldots(18)$$

$$\frac{(2p^2 - k'^2)\,\bar{\phi}'_+(p)}{Y} + 2ip\,\psi'_+(p) = \left[\frac{k\sin\theta + iY}{Y(p + k\cos\theta)} + \frac{i}{p-k\cos\theta}\right](2p^2 - k'^2). \qquad \ldots(19)$$

These equations are solved for $\bar{\psi}'_+(p)$. p is changed to $-p$ to find the value of $\bar{\psi}'_+(p) - \bar{\psi}'_+(-p)$. Then

$$\bar{\psi}_+ (p, z) - \bar{\psi}_+ (-p, z) = -\frac{1}{\delta}\left[\psi'_+ (p) - \psi'_+ (-p)\right] \exp(-\delta z)$$

$$= \frac{8p\mu k \cos\theta\,(2p^2 - k'^2)(Y - ik\sin\theta)}{(p^2 - k^2\cos^2\theta)\,F(p)} \cdot e^{-\delta z} \qquad \ldots(20)$$

where

$$F(p) = \mu\,[(2p^2 - k'^2)^2 - 4p^2 Y\delta]. \qquad \ldots(21)$$

VARIOUS WAVES

The potential function $\psi(x, z)$ is given by the inverse Fourier transform

$$\psi(x, z) = \frac{1}{2\pi} \int_{-\infty+i\beta}^{\infty+i\beta} \bar{\psi}_+ (p, z)\, e^{-ipx}\, dp,\ x > 0 \qquad \ldots(22)$$

where $-d < \beta < d$. The factor $\exp(-ipx) = \exp(i\alpha x)\ \exp(\beta x)$ in (22) vanishes as $p \rightarrow -\infty$ in the lower part of the complex plane if $x > 0$. If the contour of integration is chosen in the lower half of the complex plane, where $\bar{\psi}_+ (-p, z)$ is analytic, then

$$\psi(x, z) = \frac{1}{2\pi} \int_{-\infty+i\beta}^{\infty+i\beta} [\bar{\psi}_+ (p, z) - \bar{\psi}_+ (-p, z)]\, e^{-ipx}\, dp. \qquad \ldots(23)$$

The value of expression within the brackets is obtained in (20). (23) is evaluated along a closed contour in the lower half of the complex plane with $p = -k, -k'$ as the branch points, indentations around $p = \pm k\cos\theta$ and satisfying the conditions $\mathrm{Re}(Y) = 0 = \mathrm{Re}(\delta)$ (Fig. 2). The conditions according to Ewing *et al.*[2] imply hyperbolic paths for the branch cuts at $p = -k, -k'$. Indentations around $p = \pm k\cos\theta$ contribute the reflected transverse waves

$$D(k)\exp(-ikx\cos\theta - \delta' z) \text{ and } -D(k)\exp(ikx\cos\theta - \delta' z) \qquad \ldots(24)$$

where

$$D(k) = 4\mu k^2 \sin\theta\cos\theta\,(2k^2\cos^2\theta - k'^2)/F(k\cos\theta) \qquad \ldots(25)$$

and $\delta' = (k^2\cos^2\theta - k'^2)^{1/2}$. These waves cancel each other on the rigid boundary. The compressional waves reflected from the boundaries are

$$-\tfrac{1}{2}\exp[-ik(x\cos\theta\ z\sin\theta)] \text{ and } -\tfrac{1}{2}\exp[ik(x\cos\theta - z\sin\theta)]. \qquad \ldots(26)$$

These waves together with the incident wave vanish on the rigid boundary satisfying the boundary conditions (8).

The scattered waves are the contributions of the integrals along the branch cuts. Along the branch cut at $p = -k$, real part of Y is zero and imaginary part of Y has

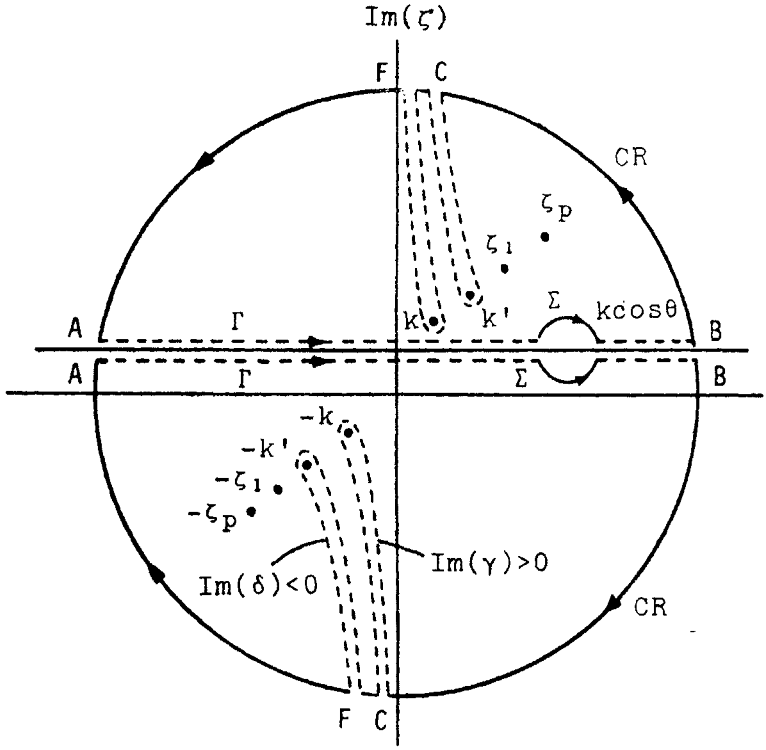

FIG. 2. Contour of integration in the complex P-plane.

opposite signs along the two sides of the branch cut. δ is unchanged along the cut. The integrals cancel each other along the branch cut at $p = -k$. Thus the scattered waves are not of the compressional type. For the contribution along the branch cut at $p = -k'$, let us take $p = -k' - iu$, u being small, then

$$\delta^2 = (-k' - iu)^2 - k'^2 = 2iu\,(k_1' + ik_2') - u^2 \qquad \ldots(27)$$

Along the cut, real $(\delta) = 0$ and δ^2 is real and negative. Thus $\delta = \pm i\delta'$, $\delta' = (2k_2' u)^{1/2}$, $k_1' = 0$ and u varies from 0 to ∞ along the cut. Evaluating (23) along the sides of the branch cut to find

$$\psi_1(x, z) = \frac{i e^{-k_2' x}}{2\pi} \int_0^\infty \Big[[\bar{\psi}_+(p, z) - \bar{\psi}_+(-p, z)]_{\delta = i\delta'}$$

$$- [\bar{\psi}_+(p, z) - \bar{\psi}_+(-p, z)]_{\delta = -i\delta'} \Big] e^{-ux}\, du \qquad \ldots(28)$$

$$\int_0^\infty \left[M(u) \sin\left(\sqrt{2k_2' u}\, z\right) + N(u) \sqrt{u} \cos\left(\sqrt{2k_2' u}\, z\right)\right] e^{-(k_2' + u)x} du \quad ...(29)$$

where

$$M(u) = \frac{8pk \cos\theta\,(Y - ik \sin\theta)\,\mu^2 (2p^2 - k'^2)^3}{\pi H(p)(p^2 - k^2\cos^2\theta)} \quad ...(30)$$

$$N(u) = -\frac{32\beta^3 Yk \cos\theta\,(Y - ik\sin\theta)(2k_2')^{1/2}\mu^2(2p^2 - k'^2)}{\pi H(p)(p^2 - k^2\cos^2\theta} \quad ...(31)$$

$$H(p) = \mu^2 [(2p^2 - k'^2)^4 + 16p^4 Y^2 \delta^2]. \quad ...(32)$$

Since u is small, $M(0)$ and $N(0)$ are retained. Following Laplace integrals are used for evaluation of (29) Oberhettinger[10] :

$$\int_0^\infty \sin\left(\sqrt{2k_2' u}\, z\right) e^{-ux} du = \frac{\sqrt{2k_2' \pi}}{2x^{3/2}} \exp\left(-\frac{k_2' z^2}{2x}\right) \quad ...(33)$$

$$\int_0^\infty \sqrt{u} \cos\left(\sqrt{2k_2' u}\, z\right) e^{-ux} du = \frac{\sqrt{\pi}(x - k_2' z^2)}{2x^{5/2}} \exp\left(-\frac{k_2' z^2}{2x}\right). \quad ...(34)$$

Further, when $x >> z$, then

$$r = (x^2 + z^2)^{1/2} = x(1 + z^2/x^2)^{1/2} = x + z^2/2x. \quad ...(35)$$

From (39), it is obtained that

$$\psi_1(x, z) = \frac{\sqrt{\pi}\, e^{-k_2' r}}{2\sqrt{r}} \left[M(0) \frac{\sqrt{2k_2'} \sin\beta}{\cos^{3/2}\beta} + N(0) \frac{\cos\beta - k_2' r \sin^2\beta}{r \cos^{5/2}\beta} \right] \quad ...(36)$$

where $x = r\cos\beta$, $z = r\sin\beta$. Thus the scattered waves in (36) are transverse waves behaving as cylindrical waves. Their amplitude is of the form $\exp(-k_2' r)/\sqrt{r}$ which decays exponentially as the distance r from the scatterer at (0,0) increases. In the free surface ($z=0$), it can be seen that (33) and (34) behave as $O(x^{-3/2})$. Close to the scatterer as $r \to 0$, the amplitude of the scattered wave behaves as $O(r^{-3/2})$.

CONCLUSIONS

It is interesting to note that the scattered waves are transverse and not compressional. For the far-field when $x >> z$, the amplitude of the scattered wave behaves as exp $(-k_2' r)/\sqrt{r}$, $r = (x^2 + z^2)^{1/2}$, r being the distance from the scatterer at (0,0). The wave is a cylindrical wave. It dies out exponentially as it moves away from the scatterer. For the near-field as $r \to 0$, the scattered wave is of the form $O(r^{-3/2})$. It is dominant near the scatterer. The amplitude of the scattered wave is plotted versus the wave number (fig. 3). For Poisson's solids, when $\alpha = \sqrt{3}\,\beta$, $\theta = \pi/3$, the amplitude in the free surface ($z=0$) of the quarter-space falls off sharply as the product of the wave number and the distance increases slowly. The amplitud has the value 36.60 when $k_2 x = .1$ and falls to the value .025 when $k_2 x = 1.8$. There is no scattered wave when $\theta = \pi/2$, i.e., when the wave is incident on the free surface parallel to the rigid boundary. The waves in (24-25) (both transverse and compressional) reflected from the boundaries of the medium satisfy the boundary conditions. The scattered waves together the waves contributed due to the branch cut at $p = k'$ also satisfy the boundary conditions.

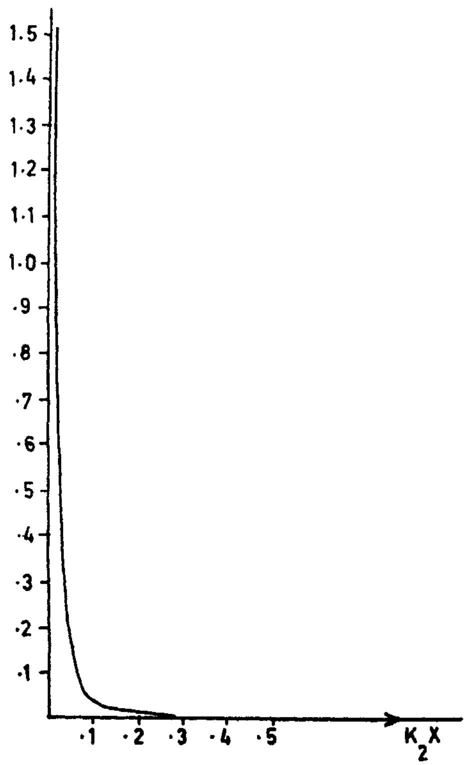

FIG. 3. Amplitude of the scattered wave.

The residues at the poles $p = \pm p_n$ contribute

$$\pm \frac{8\mu\, p_n\, k \cos\theta\, (2p_n^2 - k'^2)\,(Y_n - ik \sin\theta)}{(p_n^2 - k^2\cos^2\theta)\, F'(\pm p_n)}\, e^{-\delta_n z}\, e^{\mp i p_n x}$$

where $Y_n = (p_n^2 - k^2)^{1/2}$, $\delta_n = (p_n^2 - k'^2)^{1/2}$. These are the Rayleigh waves as SV-type waves and P-type waves. These are surface waves. Their amplitude does not depend upon the wave number and is a constant multiple of the amplitude of the incident wave.

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