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ON AUTOMORPHISMS OF FREE GROUPS

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In this paper, we study free groups and the automorphism of free groups in connection with a graph. Basic definitions and some results are given. The relevant main theorem "A graph satisfying conditions G1-3 of Gersten gives a based graph satisfying (2′)" is proved.

Introduction

In sections one, and two the basic definitions and some results on graphs and free groups are given. Also, we give the relation between an automorphism of a free group and a graph. In section three, the conditions of based graphs and the definition of the minimal based graph are given.

The main future of this work is to prove that a based graph satisfying conditions G1-3 of Gersten[2] gives a based graph satisfying (2′) which we consider it to be a generalization of Abdel-Gawad[1].

1. Graphs

1.1. A graph Γ consists of two disjoint sets, a non-empty set of vertices $V(\Gamma)$ and a set of edges $E(\Gamma)$, together with a function $t : E(\Gamma) \to E(\Gamma)$ denoted by $e \to \bar{e} \neq e$ (so $\bar{\bar{e}} = e$). Let $i(e) = t(\bar{e})$, where $i(e)$ is called the initial point of e and $t(\bar{e})$ the terminal point of $\bar{e}$ ($\bar{e}$ is called the inverse of e).

If e is in $E(\Gamma)$ and $i(e) = u$; $t(e) = v$, we say e joins u and v and write $e : u \to v$.

1.2. Apath p in the graph Γ is an ordered n-tuple $(n \geqslant 1)$ $p = (e_1, \ldots, e_n)$; $e_i \in E(\Gamma)$; $i = 1, \ldots, n$; such that $t(e_j) = i(e_{j+1})$; $1 \leqslant j \leqslant n-1$; $i(e_1)$ and $t(e_n)$ are called the initial and terminal points of p, written $i(p)$ and $t(p)$ respectively ($\bar{p} = (\bar{e}_n, \ldots, \bar{e}_1)$).

If two paths $p = (e_1, \ldots, e_n)$ and $p' = (e'_1, \ldots, e'_m)$ are such that $i(p') = t(p)$, then their composite $p.p'$ is defined by $p.p' = (e_1, \ldots, e_n, e'_1, \ldots, e'_m)$.

1.3. An elementary reduction $p \searrow p'$ of paths is a reduction by which $p = (e_1, \ldots, e_n)$ is replaced by

$p' = (e_1, ..., e_{j-1}, e_{j+2}, ..., e_n)$ if $e_{j+1} = \bar{e}_j$.

1.4. The path p is called reduced if it admits no elementary reductions.

1.5. Two paths p and p' are called equivalent, written $p \sim p'$ if there is a finite sequence of paths $p = p_1, ..., p_m = p'$ such that either $p_j \curvearrowright p_{j+1}$ or $p_{j+1} \curvearrowright p_j$ for $1 \leqslant j \leqslant m-1$.

Proposition 1.1— (a) Each path is equivalent to a unique reduced path; (b) the operation of composition of paths is consistent with equivalence[2,3,9,10].

1.6. If p_1 and p_2 are reduced paths, then $p_1 p_2$ denotes the reduced path equivalent to p_1, p_2, if defined.

1.7 A morphism[2,3] of graphs $f : \Gamma \to \Gamma'$ is a map f taking each vertex to aertex, and each edge to an edge or a vertex such that $f(\bar{e}) = \overline{f(e)}$ and $t(f(e)) = f(t(e))$ where $\bar{v} = t(v) = v$.

1.8. A graph is called connected[9] if every pair of its vertices is joined by some path. In general, a graph is the disjoint union of its connected components.

1.9. A loop at a vertex v is a path p not necessarily reduced, with $i(p) = t(p) = v$.

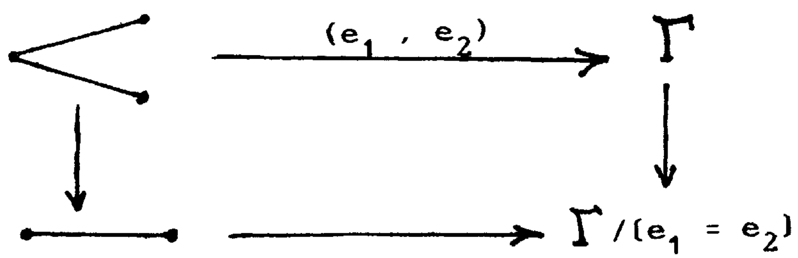
1.10. If the loop at a vertex v is reduced path, it will be called circuit.

1.11. A tree Serre[8] is a connected non-empty graph without circuits.

1.12. Let Γ' be a subgraph of a connected graph Γ. W say that Γ' is spanning if every pair of vertices of Γ is joined by at least one path Γ'.

1.13. A subgraph Γ' is a spanning tree or maximal subtree of Γ if Γ' is a tree and spanning.

1.14. A pair of edges (e_1, e_2) of Γ is said to be admissible Stallings[9] if $i(e_1) = i(e_2)$ and $e_1 \neq \bar{e}_2$. In this case we can identify $t(e_1)$ to $t(e_2)$, e_1 to e_2, $\bar{e}_1$ to $\bar{e}_2$ to obtain a graph denoted by $\Gamma/[e_1 = e_2]$. The morphism $\Gamma \to \Gamma/[e_1 = e_2]$ is called an edge fold.



1.15. Let Γ, Γ' be two graphs, an edge collapse is a morphism $f : \Gamma \to \Gamma'$ whose effect is to identify e, $\bar{e}$, $i(e)$ and $t(e)$ to one vertex.

2. Free Groups and Graphs Representing Automorphisms of Free Groups

2.1. *Free Groups*

Let $X = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ be a finite set whose elements are called letters. We define an equivalence ($\sim$) on words in X generated by $w_1 x_i^{\epsilon} x_i^{-\epsilon} w_2 \sim w_1 w_2$; $i = 1, \ldots, n$; $\epsilon = \pm 1$, where w_1, w_2 are arbitrary words in X, and $(x^{-1})^{-1} = x^{-\epsilon}$, where x_i is called the inverse of x_i^{ϵ}; $\epsilon = \pm 1$. If $w = x_{i_1}^{\epsilon_{im}}, \ldots, x_{i_m}^{\epsilon_{im}}$ then w^{-1} is defined to be $x_{i_m}^{-\epsilon_{im}} \ldots x_{i_1}^{-\epsilon_{i1}}$.

The operation of juxtaposition is compatible with equivalence. Therefore if the equivalence class of w is denoted by $[w]$ then the operation on the classes is well defined.

$$[w].\,[w'] = [w.\,w'] \qquad \ldots(1)$$

where the R.H.S. product is juxtaposition.

$[w].\,[w^{-1}] = [w^{-1}].\,[w] = [w.w^{-1}] = [w^{-1}.w] = [e]$, where e is the empty word.

Definition 2.1.1—The free group F_X on X is formed by taking as elements all equivalence classes of words in X and as group product, the operation defined by (1).

Definition 2.1.2—A word in X is said to be reduced or freely reduced if it contains no consecutive pairs of the form $x_i^{\varepsilon} \; x_i^{-\varepsilon}$; $\epsilon = \pm 1$; $i = 1, \ldots, n$.

Lemma 2.1.1—Any equivalence class contains one and only one, reduced word[4].

Henceforth the elements of F_X will be considered as reduced words.

2.2. *Automorphisms of Free Groups*

Let X and F_X be defined. We denote by Aut F_X, the automorphism group of the free group F_X.

Automorphisms of type (1) Lyndon[7] are those which merely premute the letters in X subject to the restriction that if $x_i \to x_j$, then $x_i^{-1} \to x_j^{-1}$; $i \neq j$. Automorphisms[5] of type (2) are denoted uniquely by the symbols (A, a), where A is a proper subset of X and a, which is termed the pivot of the transformation, satisfies $a \in A$, $a^{-1} \notin A$.

(A, a) is defined on F_X by mapping each letter as follows $(A, a)\, a = a^{-1}$ and $(A, a)\; a^{-1} = a$, and for $x \in X$, $x \neq a, a^{-1}$ we have

$$(A, a)x = \begin{cases} a^{-1} x a & ; \; x, x^{-1} \in A \\ xa & ; \; x \notin A, x^{-1} \in A \\ a^{-1} x & ; \; x \in A, x^{-1} \notin A \\ x & ; \; x, x^{-1} \notin A. \end{cases}$$

For $\alpha \notin X$, define $(A, \alpha) : F_X \to F_{X_{\cup\alpha}} \pm 1$ as a monomorphism given by

$$(A, \alpha)\ x = \begin{cases} \alpha^{-1} x\alpha & ;\ x, x^{-1} \in A \\ x\alpha & ;\ x \notin A, x^{-1} \in A \\ \alpha^{-1}x & ;\ \alpha \in A, x^{-1} \notin A \\ x & ;\ x, x^{-1} \notin A. \end{cases}$$

This is a paraphase of Abdel-Gawad[1].

Definition 2.2.1—A labelling is a map $f : E(\Gamma) \rightarrow F_Y$, such that $f(\bar{e}) = f(e)^{-1}$ for all $e \in E(\Gamma)$. If $p = e_1 \ldots e_n$ is a path in Γ then $f(p) = f(e_1) \ldots Xf(e_n)$.

Definition 2.2.2—The based graph Γ which represets (A, α) has two vertices A and A' where A' (the complement of A in X) is the base point, and a directed edge labelled $x^{\pm 1}$, for each $x^{\pm 1} \in X$ and for α, α^{-1} such that $i(x) \in A \Rightarrow x \in A$, $i(\alpha) \in A$, $t(\alpha) \in A'$. Call this labelling f; as example, let $A = \{x_1, x_2^{\pm 1}, x_3^{-1}\} \subset X = \{x_1^{\pm 1} \ldots, x_4^{\pm}\}$.

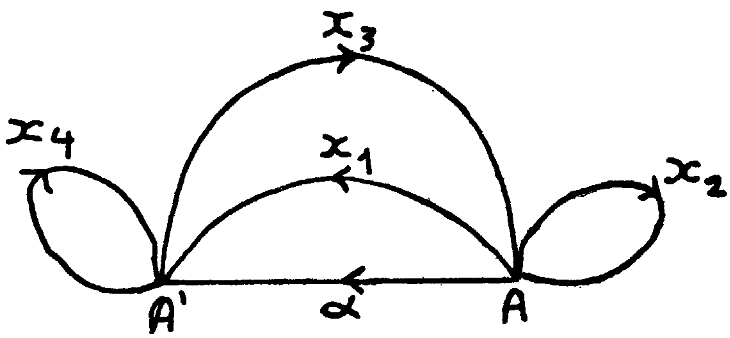


FIG. 1.

We now show how to construct from this graph Γ a graph representing (A, a). Define two labellings on Γ as follows. The first label f_1 of the edge labelled α is 1 and the other edges are the same as f. The second label f_2 is 1 on the directed edge with first label a, is a on the directed edge with first label 1, i.e. on edge labelled α in Γ, and is the same as the first label on the other edges.

Henceforth, we use (x, y) to denote on edge with first label x and second label y.

Proposition—For each $w \in Fx$ there exists a unique circuit p at A' such that $f_1(p) = w$. Moreover $f(p) = (A, \alpha)\, w$ and $f_2(p) = (A, a)\, w$.

PROOF : The result is clear for $w = x^{\pm 1}$ and is proved for arbitrary w by induction.

From the last example, if $a = x_1$; then

$(A, a) = (\alpha = 1).\ (x_1 \Leftrightarrow \alpha)\ (A, a)$ and $\alpha \rightarrow x_1$, $x_1 \rightarrow 1$, $x_2 \rightarrow x_2$, $x_3 \rightarrow x_3$, $x_4 \rightarrow x_4$.

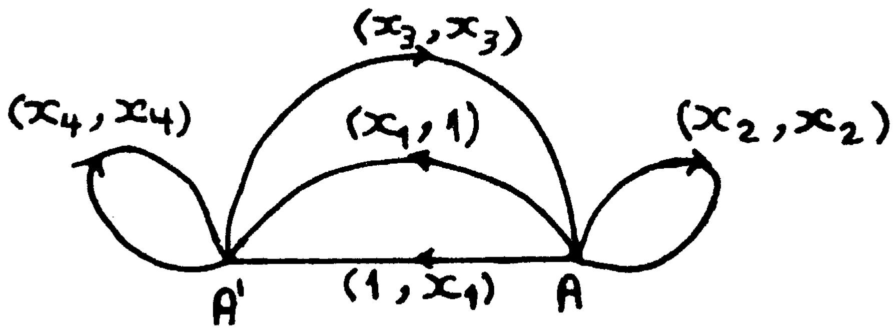


FIG. 2.

3. THE USE OF BASED GRAPHS TO REPRESENT AUTOMORPHISMS

Let Γ be a based graph with two labellings f_1 and f_2 in F_X satisfying the conditions :

1. The labels of any edge are either in X or the identity, and every $x \in X$, is the first label of some edge, and the second label of some edge.

2. T_1 and T_2 are spanning, where T_1 is the set of all edges with first label 1, T_2 is the set of all edges with second label 1

3. No two edges have the same initial vertex, first and second label. (other than (1.1)).

4. Given edges e, e' both having labels $(x, 1)$; $x \in X$, then any path in T_1 from $i(e)$ to $i(e')$ has the same second label as any path in T_1 from $t(e)$ to $t(e')$. Similarly, if e and e' both have labels $(1, x)$; $x \in X$, then any path in T_2 from $i(e)$ to $i(e')$ has the same first label as any path in T_2 from $t(e)$ to $t(e')$.

5. For every $x \in X$, all edges with second label x dave the same second label and similarly, all edges with second label x have the same first label.

6. Any loop in T_1 has second label equal to 1 in Fx, similarly any loop in T_2 has first label equal to 1 in Fx.

Let v_0 be the base point of Γ.

Lemma 3.1—For every reduced word $x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}$; $\epsilon_i \pm 1$, $i = 1, \ldots, n$ there is a reduced path p such that $i(p) = v_0$, $t(p) = v_0$ and $f_1(p) = x_1^{\epsilon_l} \ldots x_n^{\epsilon_n}$.

Lemma 3.2—All paths with first label $x_1^{\epsilon_1} \ldots x_n^{\epsilon_n} \in Fx$ and $i(p) = v_0 = t(p)$ have the same second label.

From Lemmas 3.1 and 3.2,, we can define a map $\phi : Fx \to Fx$ on reduced words in X by :

$$\phi (x_1^{\epsilon_1} \dots x_n^{\epsilon_n}) ; \epsilon_i = \pm 1, i=1,\dots, n;$$

equals the reduced word equivalent to the second label of any path p ($i(p) = t(p) = v_0$) with first label $x_1^{\epsilon_1} \dots, x_n^{\epsilon_n}$.

Similarly, we can define $\psi : Fx \to F_X$ by

$$\psi (x_1^{\epsilon_1} \dots x_n^{\epsilon_n}); \epsilon_i = \pm 1, i = 1, \dots, n;$$

equals the reduced word equivalent to the first label of any path p ($i(p) = t(p) = v_0$) with second label $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$.

Lemma 3.3—ϕ is a homomorphism.

Theorem 3.1—ϕ and ψ are inverses of each other, and hence they are automorphisms.

Definition 3.1—If (Γ, v_0) is a based graph satisfying the conditions 1—6, then (Γ, v_0) is called a based graph representing ϕ.

Corollary 3.1—Every automorphism is represented by a based graph (satisfying 1—6).

Definition 3.2—(1) If v is a vertex in (Γ, v_0) with valency 2, $v \neq v_0$ and if (i) v is an extreme vertex of both T_1 and T_2; (ii) the labels $(x, 1)$ and $(1, y)$ of the two edges in $\text{Star}_\Gamma(v) = \{e \in (\Gamma, v_0)/i(e) = v\}$ occur as the labels of some other edges of (Γ, v_0), then v and its adjacent edges are redundant.

(2) If $v = v_0$ satisfies condition (i) but not (ii) then we say that v and its adjacent edges are singular.

Definition 3.3—From a based graph, we get a smaller graph as follows :

M1. Identify vertices by edge collapses and edge folds, collapsing all edges labelled (1, 1) and folding all edges with both labels the same and with the same initial vertex, until no further such folds are possible.

M2. Remove, all the redundant vertices, and their adjacent edges.

M3. After removing all redundant vertices, combine every two edges e_1 and e_2 say, where e_1 is an edge labelled $(x, 1) \in T_2$ with $i(e_1) = u$, $t(e_1) = v$ and e_2 is an edge labelled $(1, y) \in T_1$ with $i(e_2) = v$, $t(e_2) = u_1$ into one edge e labelled (x, y) with $i(e) = u$, $t(e) = u_1$ whenever v is singular.

Observation : The operation M2 and M3 do not introduce any edge to which M1 can be applied or any redundant vertices.

Definition 3.4—A based graph with,

(i) No edge labelled (1, 1),

(ii) No possible edge fold (for edges with same labels),

(iii) No redundant vertices,

(iv) No singular vertices,

is called a minimal based graph.

Proposition 3.1—After applying M1 to the graph (Γ, v) T_1 and T_2 are disjoint and spanning; every vertex v with valency 2 is either singular or redundant; if v is singular then neither of the labels $(x, 1)$ and $(1, y)$ of the two edges in $\mathrm{Star}_\Gamma(v)$ occurs as the label of any other edges of (Γ, v_0).

Theorem 3.2—After applying M1, M2 and M3 to a based graph Γ. We still have a based graph.

Theorem 3.3—In a minimal based graph the following condition holds (2′) T_1 and T_2 are disjoint maximal trees.

4. The main Result

Definition 4.1[2,3,9,10]—If Γ is a graph and $v \in V(\Gamma)$ let $\mathrm{Star}_\Gamma(v) = (e \in \Gamma / i(e) = v)$ A morphism $f : \Gamma \to \Gamma'$ of graphs induces a map

$f_v : \mathrm{Star}_\Gamma(v) \to \mathrm{Star}_{\Gamma'}(f(v))$. We say f is an immersion if f_v is injective for each $v \in V(\Gamma)$.

Definition 4.2[2]—A morphism $f : \Gamma \to \Gamma'$ of finite graphs is called special if it is a composition of edge collapses and edge folds.

Lemma 4.1[1]—Let Y be a finite graph with one vertex y_0 labelled $1 \in Fx$, and one edge labelled $x^{\pm 1}$ for each $x \in X$. Let l denote the labelling of $Y \to Fx$. For each labelling f of $E(\Gamma) \to F_X$ satisfying (1) (page 7) extend f by mapping every vertex to $l \in Fx$. Then there exists a unique graph morphism $\tilde{f}$ of $\Gamma \to Y$ such that $l.\tilde{f} = f$.

Theorem 4.1[1]—Let (Γ, v_0) be a based graph satisfying (2′) and let $\tilde{f}_1$ and $\tilde{f}_2$ be defined, and let ϕ be the automorphism of Fx represented by Γ. Then the following conditions Gersten[2] holds :

G1. T_1 and T_2 are maximal trees of Γ and $E(T_1) \cap E(T_2) = \phi$,

G2. $\tilde{f}_1 / (\Gamma - E(T_1))$ and $\tilde{f}_2 / (\Gamma - E(T_2))$ are both immersions, G3. $\tilde{f}_1$ and $\tilde{f}_2$ are special, $\ker \tilde{f}_1^* = \ker \tilde{f}_2^*$ and $\tilde{f}_2^* \cdot \tilde{f}_1^{-1*} = \tilde{\phi}$, where $\tilde{f}_1^*, \tilde{f}_2^* : \pi_1(\Gamma, v) - \pi_1(Y, y_0)$; $\pi_1(\Gamma, v)$ is the set of equivalence classes of paths p such that $i(p) = t(p)\, v$, $\tilde{\phi}$ is the automorphisms of $\pi_1(Y, y_0)$ corresponding to ϕ.

Now we will prove the converse to Theorem 4.1.

Theorem —A graph satisfying conditions G1—3 of Gersten[2] gives a based graph satisfying (2′).

PROOF : G1 is precisely (2′). Also from G1, since T_1 and T_2 are maximal disjoint trees, then any loop in T_1 at the vertex v has second lebel which equal to 1 in Fx. Otherwise contradicting G1. Similarly, any loop in T_2 has first label which equal to 1 in F_X. Then (6) holds.

From G2, since $f_i/(\Gamma - E(T_i))$; $i = 1, 2$ is an immersion, then f_i is injective on $\text{Star}_\Gamma(v)/T_i$ which means that (3) holds.

Now to prove that (4) and (5) hold, iet (Γ, v_0) be a based graph with two labellings f_1 and f_2, and consider the following assumptions :

(a) T_1 is spanning tree,

(b) $f_1(p) = 1 \Rightarrow f_2(p) = 1$ whenever p is a circuit at v_0 Hoare,

(c) $f_2(p)$ is reduced whenever p is a reduced path with no edges in T_2 Hoare,

(d) $T_1 \cap T_2 = \phi$.

Then we prove that (Γ, v_0) as the following property: G′4. If e and e' are two distinct edges with $f_1(e) = f_2(e') = 1$, then $e, e' \in T_2$. Moreover T_1 is a tree and if α and β are the unique paths in T_1 from $i(e)$ to $i(e')$ and $t(e)$ to $t(e')$ respectively, then $f_2(\alpha) = f_2(\beta)$. To do so, if T_1 contains a non-trivial circuit p then using (a), T_1 contains a non-trivial reduced circuit p at v_0. From assumption (d), p has no edges in T_2, and on using (c) and (b), $f_2(p)$ is reduced with $f_2(p) = 1$, so p is empty, giving a contradiction, therefore T_1 must be a tree.

Now suppose that $e \in T_2$, and let p be the circuit $q.\, e'\, \bar{\beta}.\bar{e}.\alpha.\, \bar{q}$ as in Fig. 3, where q is the unique path in T_1 from v_0 to $i(e')$ (since T_1 spanning tree). Then $f_1(p) = f_1(e')\, f_1(e) = 1$, and on using assumption (b),

$$f_2(p) = f_2(q)\, f_2(e')\, f_2(\bar{\beta})\, f_2(\bar{e})\, f_2(\alpha)\, f_2(\bar{q}) = 1. \qquad \ldots(1)$$

According to (d) $\bar{\beta}.\bar{e}.\alpha$ has no edges in T_2, therefore by (c) $f_2(\bar{\beta}.\bar{e}.\alpha)$ is reduced and

Fig. (3)

from (1) $f_2(\overline{\beta.\bar{e}.\alpha}) = f_2(e)^{-1}$. Thus α and β are empty and $f_2(e) = f_2(e') = 1$. Then $e'.e$ is a reduced path which contradicts (c). Thus $e \in T_2$. Similarly $e' \in T_2$, and since $f_2(\beta)$ and $f_2(\alpha)$ are reduced according to (c) with $f_2(e'.(e'.\bar{\beta}.\bar{e}.\alpha) = 1$, then $f_2(\beta) = f_2(\alpha)$.

Similarly, on taking the assumptions;

(a') T_2 is spanning tree,

(b') $f_2(p) = 1 \Rightarrow f_1(p) = 1$, whenever p is a circuit at v_0,

(c') $f_1(p)$ is reduced whenever p is a reduced path with no edges in T_1,

(d') $T_1 \cap T_2 = \phi$;

and applying the same arguments as in the proof of G'4, one can prove that (Γ, v_0) has the following property; if e and e' are two distinct edges with $f_2(e) = f_2(e') = 1$, then $e, e' \in T_1$, moreover T_2 is a tree and if α and β are the unique paths in T_2 from $i(e)$ to $i(e')$ and $t(e)$ to $t(e')$ respectively, then $f_1(\alpha) = f_1(\beta)$. This completes the proof of the theorem.

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