Indian J. pure appl. Math., 20 (1) : 58-63, January 1989

A GENERALIZATION OF STRONGLY REGULAR NEAR-RINGS

P. Dheena

Department of Mathematics, Annamalai University
Annamalainagar 608 002

(*Received* 11 *May* 1987; *after revision* 9 *May* 1988)

In this paper we introduce the notion of s-weakly regular near-rings similar to the notion introduced for rings by Gupta[1]. We give some characterizations of s-weakly regular near-rings.

Definitions and Notations

Throughout this paper N stands for a zerosymmetric right near-ring. For any $x \in N$, $<x>$ stands for the principal ideal generated by x. N is said to be s-weakly regular if for each $a \in N$, $a = xa$ for some $x \in <a^2>$.

For any subset S of N, we denote the set $\{n \in N : nS = 0\}$ by $A(S)$. We denote $A(\{a\})$ by $A(a)$. An ideal P of N is called completely prime (semicompletely prime) if $ab \in P$ implies $a \in P$ or $b \in P$ ($a^2 \in P$ implies $a \in P$). An ideal I of N is said to be prime if for any two ideals A, B of N, $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$. This is equivalent to the following condition. For all finite set $A_1, A_2 \ldots. A_n$ of ideals of N, $A_1 A_2 \ldots. A_n \subseteq I$ implies atleast one $A_i \subseteq I$. An ideal minimal in the set of all prime ideals of N is called minimal prime ideal in N. An ideal I of N is called semiprime if for any ideal J of N, $J^2 \subseteq I$ implies $J \subseteq I$. A subset M of N is called an m-system if for any a, b in M, there exists $a_1 \in <a>$ and $b_1 \in <b>$ such that $a_1 b_1 \in M$. N is said to have IFP (insertion of factors property) if for a, b in N, $ab = 0$ implies $anb = 0$ for all $n \in N$. N is said to be reduced if it is without nonzero nilpotent elements. If N is a zerosymmetric reduced near-ring, then by Pilz[4] (p. 290), N has IFP and for any x, y in N, $xy = 0$ implies $yx = 0$. For the basic terminology and notation we refer to Pilz[4].

Section 1

Lemma 1— If N is a reduced near-ring, then for any $0 \neq a \in N$,

(1) $A(a)$ is a semicompletely prime ideal,

(2) $N/A(a)$ is reduced and the residue class $\bar{a}$ of a mod $A(a)$ is a nonzero divisor,

(3) $x_1 x_2 \dots x_n = 0$ implies $\langle x_1 \rangle \langle x_2 \rangle \dots \langle x_n \rangle = 0$

for any $x_1, x_2, \dots, x_n$ in N.

Proof: (1) Since N has IFP, $A(a)$ is an ideal. Suppose $x^2 \in A(a)$. Then $0 = x^2 a = x(xa) = xax$, so $(xa)^2 = 0$. Thus $xa = 0$ and hence $A(a)$ is a semicompletely prime ideal.

(2) Since $A(a)$ is a semicompletely prime ideal, $N/A(a)$ is reduced.

Suppose there exists $b \in N$ with $\bar{b}\,\bar{a} = \bar{0}$. Then $\bar{a}\,\bar{b} = \bar{0}$ and hence $ab \in A(a)$. Thus $(ba)^2 = baba = 0$, so $ba = 0$ and hence $\bar{b} = \bar{0}$.

(3) Suppose $x_1 x_2 \dots x_n = 0$ for some $x_1, x_2, \dots, x_n$ in N. Since N is reduced, $A(S)$ is an ideal for any subset S of N. Since $x_1 \in A(x_2 \dots x_n)$, we have $\langle x_1 \rangle \subseteq A(x_2 \dots x_n)$ so that $\langle x_1 \rangle x_2 \dots x_n = 0$. Hence $x_2 \dots x_n \langle x_1 \rangle = 0$. Since $x_2 \in A(x_3 \dots x_n \langle x_1 \rangle)$ we have $\langle x_2 \rangle \subseteq A(x_3 \dots x_n \langle x_1 \rangle)$ so that $\langle x_2 \rangle x_3 \dots x_n \langle x_1 \rangle = 0$. Hence $x_3 \dots x_n \langle x_1 \rangle \langle x_2 \rangle = 0$. Continuing this process we arrive at (3).

Theorem 1 — Let N be a reduced near-ring. If M a subset of N, is a non-void m-system such that $0 \notin M$, then there exists a completely prime ideal P of N such that $P \cap M = \phi$.

Proof: Let M' be a maximal m-system relative to the property $M \subseteq M'$ and $0 \notin M$. M' is obtained using Zorn's Lemma and clearly $M \subseteq M'$. By Pilz[4] (Proposition 2.81) there exists a prime ideal $P\ (\neq N)$ such that $P \cap M' = \phi$. Hence $M' \subseteq N \setminus P$. By Pilz[4] (Corollary 2.80) $N \setminus P$ is an m-system. By the maximality of M', $N \setminus P \subseteq M'$. Thus $N \setminus M' = P$. It can be easily verified that P is a minimal prime ideal in N. Now let us show that P is completely prime. Let $\bar{M}$ be the multiplicative subsemigroup of N generated by $N \setminus P$. We claim that $0 \notin \bar{M}$. If not, there exists $x_1, x_2, \dots, x_n \in N \setminus P$ such that $x_1 x_2 \dots x_n = 0$. By Lemma 1, $\langle x_1 \rangle \langle x_2 \rangle \dots \langle x_n \rangle = 0 \subseteq P$. Thus $\langle x_i \rangle \subseteq P$ for some i. Hence $x_i \in P$ which contradicts our assumption. Let $K = \{J : J$ is an ideal of N such that $J \cap \bar{M} = \phi\}$. K is not empty. By Zorn's Lemma, K contains a maximal element, say Q. Now we claim Q is prime. For, if there exists ideals A, B such that $Q \subset A$ and $Q \subset B$ then take $a \in A \cap \bar{M}$ and $b \in B \cap \bar{M}$. Then $ab \in \bar{M}$ and $ab \in \langle a \rangle \langle b \rangle \subseteq AB$. Hence $\langle A\, B \rangle \cap \bar{M} \neq \phi$ so that $\langle A\, B \rangle \not\subseteq Q$ and $A\, B \not\subseteq Q$. Thus Pilz[4] (Proposition 2.61) Q is prime. Now $Q \subseteq N \setminus \bar{M} \subseteq P$. Since P is a minimal prime ideal $Q = N \setminus \bar{M} = P$. Since $\bar{M}$ is a semi-group, P is completely prime.

Corollary 1 — Let N be a reduced near-ring. If M is a non-void multiplicative subsemigroup of N such that $0 \notin M$, then there exists a completely prime ideal P of N such that $P \cap M = \phi$.

PROOF : Since every multiplicative subsemigroup is an m-system, the corollary follows immediately.

Now we prove our main theorem.

Theorem 2— The following are equivalent for a near-ring N with identity :

(1) N is s-weakly regular,

(2) N is reduced and every proper prime ideal is maximal,

(3) N is reduced and every proper completely prime ideal is maximal.

PROOF : (1) $\Rightarrow$ (2)— Suppose $a \in N$ such that $a^2 = 0$. We have $a = xa$ for some $x \in < a^2 > = 0$, so that $a = 0$. Thus $a^2 = 0$ implies $a = 0$ for every a in N. Hence N is reduced. Let P be a proper prime ideal and suppose P is properly contained in a maximal ideal M. Let $x \in M \setminus P$. Then $x = yx$ for some $y \in < x^2 >$. Now for any $n \in N$, $nx = nyx$ so that $(n-ny)\ x = 0$. By Lemma 1, $< (n-ny) > < x > = 0 \subseteq P$. Since P is a prime ideal and $x \notin P$, we have $n-ny \in P \subset M$. Further $y \in < x^2 > \subseteq M$ whence $ny \in M$ so that $n \in M$. Hence $M = N$, a contradiction.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1)— Let $0 \neq a \in N$. By Lemma 1, $\bar{N} = N/A(a)$ is reduced and $\bar{a}$ is not a zero divisor. Also every proper completely prime ideal of $\bar{N}$ is a maximal ideal in $\bar{N}$. Now let M be the multiplicative semigroup generated by all elements $\bar{a} - \bar{x}\,\bar{a}$ where $x \in < a^2 >$. We claim that $\bar{0} \in M$. If not, by Corollary 1, there exists a completely prime ideal $\bar{P}$ with $\bar{P} \cap M = \phi$. Suppose $< \bar{a}^2 > \subseteq \bar{P}$, then $\bar{a}^2 \in \bar{P}$ whence $\bar{a} \in \bar{P}$. Now for any $x \in < a^2 >$, $\bar{a} - \bar{x}\ \bar{a} \in \bar{P} \cap M$, a contradiction. Suppose $< \bar{a}^2 > \nsubseteq \bar{P}$. Since $\bar{P}$ is maximal, $\bar{P} + < \bar{a}^2 > = \bar{N}$. Hence $\bar{1} = \bar{\alpha} + \bar{x}$ for some $\bar{\alpha} \in \bar{P}$ and for some $\bar{x} \in <\bar{a}^2>$. So $\bar{1} - \bar{x} = \bar{\alpha}$ and hence $\bar{a} - \bar{x}\bar{a} = \bar{\alpha}\bar{a} \in \bar{P} \subset M$, a contradiction, Thus $\bar{0} \in M$. Now $\bar{0} = (\bar{a} - \bar{x}_1\,\bar{a}) \ldots (\bar{a} - \bar{x}_n\,\bar{a})$ where $x_i \in < a^2 >$. Since $\bar{N}$ is reduced and $\bar{a}$ is not a zero divisor, we have $(\bar{1} - \bar{x}_1)\ (\bar{1} - \bar{x}_2) \ldots (\bar{1} - \bar{x}_n) = \bar{0}$. It can be verified that $\bar{1} = \bar{x}$ for some $x \in < a^2 >$. Hence $(I - x) \in A(a)$. Thus $a = xa$ for some $x \in < a^2 >$.

Corollary 2 (Gupta[1], Theorem 5)— The following are equivalent for a ring A with identity :

(1) A is s-weakly regular,

(2) A is reduced and every proper prime ideal is maximal,

(3) A is reduced and every proper completely prime ideal is maximal.

PROOF : When the near-ring N is also a ring, our definition of s-weakly regular coincide with the same given by Gupta[1], since $a = xa$ if and only if $a = ax$ for some $x \in < a^2 >$.

Recall that a near-ring N is said to be strongly regular if given $a \in N$ there exists $x \in N$ such that $a = xa^2$.

Remark 1 : If N is strongly regular, then N is s-weakly regular.

PROOF : Suppose N is strongly regular. Let $a \in N$. Then $a = xa^2$ for some $x \in N$. By Reddy and Murty[5], (Theorem 3), $a = axa$ and $ax = xa$. Thus $a = (ax)axa = xa^2 xa = ya$ where $y = xa^2 x \in \langle a^2 \rangle$.

The following example shows that the existence of s-weakly regular rings that are not strongly regular.

Example 1— Consider the ring R given in Example 1 of Gupta[1].

Corollary 3— If N is a strongly regular near-ring, then every proper prime ideal is maximal.

The above corollary follows from Remark 1 and Theorem 2 in which (1) implies (2) is true even without the assumption N contains an identity. The above corollary improves the result of Mason[3] (Lemma 4) namely, if N is a strongly regular near-ring with identity then every proper prime ideal is maximal.

Recall that a near-ring N is called strict weakly regular (weakly regular), if $A^2 = A$ for every N-subgroup (ideal) A of N.

Theorem 3— If N is a reduced strict weakly regular near-ring with identity, then N is s-weakly regular.

PROOF : Suppose N is a reduced strict weakly regular near-ring. Let P be a proper completely prime ideal of N. Then $\bar{N} = N/P$ is strict weakly regular and $\bar{N}$ is without zero divisors. By Jat and Choudhary[2] (p. 179) $\bar{N}$ is simple so that P is maximal. By Theorem 2, N is s-weakly regular.

Lemma 2— If N is s-weakly regular, then every ideal of N is semicompletely prime.

PROOF : Suppose N is s-weakly regular. Let I be an ideal of N and let $a^2 \in I$. We have $a = xa$ for some $x \in \langle a^2 \rangle \subseteq I$. Thus $a \in I$.

Theorem 4— If N is an s-weakly regular near-ring, then every ideal I of N is the intersection of all maximal ideals containing I.

PROOF : Let I be an ideal of N. By Lemma 2, I is semi-completely prime and hence I is semiprime. By Sambasiva Rao[6] (Corollary 2.3) I is the intersection of all prime ideals containing I. By Theorem 2, I is the intersection of all maximal ideals containing I.

Theorem 5— Let N be s-weakly regular. Then

(1) N is weakly regular.

(2) N is a subdirect product of simple reduced near-rings.

PROOF : (1) Let I be an ideal of N and $a \in I$. Since $a = xa$ for some $x \in \langle a^2 \rangle \subseteq I$, we have $I \subseteq I^2$. Thus N is weakly regular.

(2) By Theorem 2, N is reduced. By Theorem 4, $\{O\}$ is the intersection of maximal ideals. Hence by Pilz[4] (p. 25) N is isomorphic to subdirect product of simple reduced near-ring.

SECTION 2

Now we find conditions for an s-weakly regular near-ring to be strongly regular.

Lemma 3— Let N be a reduced near-ring. For any a, b in N, if e is an idempotent in N, then $abe = aeb$.

PROOF : Since N is reduced, N has IFP and $xy = 0$ implies $yx = 0$ for any x, y in N. Let e be an idempotent in N and $a, b \in N$. Since $(a-ae)e = 0$, we have $(a-ae)be = 0$. Hence $abe = aebe$. Since $(eb-ebe)e = 0$, we have $eb\ (eb-ebe) = 0$ and $ebe\ (eb-ebe) = 0$ so that $(eb-ebe)^2 = 0$. Hence $eb = ebe$. Thus $abe = aeb$.

Theorem 6— Let N be an s-weakly regular near-ring. N is strongly regular if and only if every N-subgroup of N is an ideal.

PROOF : Suppose every N-subgroup of N is an ideal. Let $a \in N$ and $a \neq xa^2$ for all $x \in N$. Since Na^2 is an N-subgroup by Theorem 4, Na^2 is the intersection of all maximal ideals containing Na^2. Hence there exists a maximal ideal M, containing Na^2 such that $a \notin M$. But $a^3 \in M$. By Lemma 2, M is semicompletely prime. Hence $a \in M$, a contradiction. Therefore N is strongly regular.

Conversely let N be strongly regular. Since N is reduced, N has IFP. Let $0 \neq a \in N$. By Reddy and Murty[5], $a = axa$ for some $x \in N$. Let $xa = e$. Then e is an idempotent and $Na = Ne$. Denoting the set $\{n-ne : n \in N\}$ by S, we claim that $A(S) = Ne$. Since $(n-ne)\, e = 0$ for any $n \in N$, using IFP, $(n-ne)\, Ne = 0$ so that $Ne\,(n-ne) = 0$. Hence $Ne \subseteq A(S)$. Suppose $z \in A(S)$. There exists some $y \in N$ such that $z = yz^2$, Now $(yz-yze)z = 0$. Hence $z = yzez$. By Lemma 3, $z = yzze = ze$. Thus $A(S) \subseteq Ne$. Hence $Na = Ne = A(S)$. Since N has *IFP*, $A(S)$ is an ideal and hence Na is an ideal of N. If Δ is an N-subgroup, then $\Delta = \sum_{a \in \Delta} Na$. Since the sum of any family of ideals is an ideal, Δ is an ideal.

Corollary 4— (Mason[3], Theorem 1)— If N is strongly regular with identity, then every N-subgroup of N is an ideal.

For any subset A of N, we write $\sqrt{A} = \{x \in N : x^k \in A$ for some positive integer $K\}$. Recall that N is said to be regular if given $a \in N$, there exists $x \in N$ such that $a = axa$.

Theorem 7— Let N be s-weakly regular. Then the following are equivalent :

(1) N is regular,

(2) $A = \sqrt{A}$ for every N-subgroup A of N,

(3) N is strongly regular.

PROOF : (1) $\Rightarrow$ (2) — Let A be an N-subgroup of N. Suppose $a \in \sqrt{A}$. Then $a^k \in A$ for some positive integer k. Since N is s-weakly regular $a = xa$ for some $x \in < a^2 >$. Since N is regular $a = aya$ for some $y \in N$. Since ya is an idempotent, by Lemma 3, $a = xa = xa\,(ya) = xya^2 = xy(xya^2)\;a = xyxya^3 = \ldots = za^k$ for some $z \in N$. Thus $a = za^k \in A$. Hence $A = \sqrt{A}$.

(2) $\Rightarrow$ (3)— Let $0 \neq a \in N$ Now $a^3 \in Na^2$ so that $a \in \sqrt{Na^2} = Na^2$. Thus N is strongly regular.

(3) $\Rightarrow$ (1) follows from Theorem 3 of Reddy and Murty[5].

REFERENCES

1. V. Gupta, *Acta Math. Hung.* 43 (1984), 57–61.
2. J. L. Jat and S. C. Choudhary, *Math. Student* 4 (2) (1978), 175–82.
3. G. Mason, *Proc. Edinburgh Math. Soc.* 23 (1980), 27–35.
4. G. Pilz, Near-rings. North-Holland, Amsterdam, 1977.
5. Y. V. Reddy and C. V. L. N. Murty, *Proc. Edinburgh Math. Soc.* 27 (1984), 61–64.
6. V. Sambasiva Rao, *J. Aust. Math. Soc.* 32 (1984), 212–14.