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FREE TORSIONAL VIBRATION OF A NONHOMOGENEOUS SEMIINFINITE SOLID CIRCULAR CYLINER

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Free torsional vibration of a nonhomogeneous semiinfinite solid circular cylinder with stress free and rigid boundary is solved in this paper. The nonhomogeneity of rigidity μ and density is chosen in the from

$$\mu = \mu_0 (1 + KZ)^n, \ \rho = \rho_0 (1 + KZ)^n.$$

The displacement and stress component for different values of n are numerically evaluated.

INTRODUCTION

Several attempts have been made to solve the problem of torsional oscillation in a circular cylinder by Love[1], Kolsky[2], Davies[3] and many others. Forced torsional vibration in a circular cylinder was studied e. g. by Mitra[4], Banerjee[5], Campbell and Tsao[6] and Mondal[7]. In all of these problems the curved boundary of the cylinder is taken to be stress-free. In many engineening applications, however, a cover or a shield is provided as a protection. Therefore the particles on the curved boundary are constrained from moving. Such a boundary condition may be mathematically accounted for by taking it as a rigid boundary. Recently free torsional vibration in an isotropic homogeneous infinitely long solid circular cylinder with rigid boundary was considered by Rao[8].

In the present paper an attempt has been made to extend the previous problem to the case of free torsional vibration of a semiinfinite solid circular cylinder with rigid boundary when the material of the cylinder is nonhomogeneous in rigidity and density. In order to compare we have also considered the analogous case when the curved surface is free from traction. The nonhomogeneity is taken in the form

$$\mu = \mu_0 (1 + KZ)^n, \ \rho = \rho_0 (1 + KZ)^n$$

where μ_0, ρ_0, K are constants, n is a rational number, z-is the axial co-ordinate. Solution is derived by the method of separation of variables. Explicit formulae are given for the stress component and particle displacement. These formulae are used to compute the particle displacement and stress component at a fixed depth and are compared graphically. As regards the case of rigid boundary nodal cylinder, the pattern is the same as in the case of stress free boundary.

Formulation and Solution of the Problem

We consider a semiinfinite solid circular cylinder of radius 'a'. The origin is taken at the centre of one end of the cylinder, Z-axis is directed along the axis of the cylinder. We use cylindrical polar co-ordinates (r, θ, z) to specify the points of the cylinder. We first consider the case when the curved boundary surface is stress-free. It is assumed that the stress is distributed with radial symmetry so that the displacement is tangential and hence is independent of the angular co-ordinate θ. Then the displacement components are given by

$$u_r = u_Z = 0, u_\theta = u_\theta (r, Z, t). \qquad ...(1)$$

For this displacement field the only nonvanishing stress components are

$$\tau_{r\theta} = \mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \tau_{\theta Z} = \mu . \frac{\partial u_\theta}{\partial Z} . \qquad ...(2)$$

We assume the modulus of rigidity and density of the material in the form

$$\mu = \mu_0 (1 + KZ)^n, \rho = \rho_0 (1 + KZ)^n \qquad ...(3)$$

where μ_0, ρ_0, K are constants, n is a rational number.

The only nonvanishing equation of motion then reduces to

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial Z^2} + \frac{nK}{1 + KZ} \frac{\partial u_\theta}{\partial Z} = \frac{1}{\beta^2} \frac{\partial^2 u_\theta}{\partial t^2}$$

where

$$\beta^2 = \frac{\mu_0}{\rho_0} . \qquad ...(4)$$

When the motion of every particle of the body is simple harmonic and of period $\frac{2\pi}{p}$ the displacement may be expressed by

$$u_\theta = v (r, Z) e^{ipt}. \qquad ...(5)$$

When the body is vibrating freely the equation of motion and boundary condition can be satisfied only if p will be one of the roots of the frequency equation.

Substituting (5) in the above equation it takes the form

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial Z^2} + \frac{nK}{1+KZ} \frac{\partial v}{\partial Z} + \frac{p^2}{\beta^2} v = 0.$$

Using the method of separation of variables, we assume the solution of the equation in the form

$$v = R(r) F (Z) \qquad ...(6)$$

the above equation then changes to

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{R}{r^2}+\frac{p^2}{\beta^2}\right)$$

$$=-\frac{1}{F}\left(\frac{d^2F}{dZ^2}+\frac{nK}{1+KZ}\frac{dF}{dZ}\right)=-s^2$$

where s is a constant, which then splits up into two equations, namely,

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{1}{r^2}\right)R=0 \quad ...(7)$$

where

$$\lambda^2=\frac{p^2}{\beta^2}+s^2 \quad ...(8)$$

and

$$\frac{d^2F}{dZ^2}+\frac{nK}{1+KZ}\frac{dF}{dZ}-s^2F=0. \quad ...(9)$$

The solution of (8) is

$$R=C\,J_1(\lambda r)+DY_1(\lambda r)$$

where C, D are constants and J_1, Y_1 are Bessel functions of the first and second kind respectively of order unity.

Now for a solid cylinder, $\frac{u_\theta}{r}$ must be finite at $r=0$,

hence $D=0$ and there by $R=C\,J_1(\lambda r)$. ...(10)

To solve eqn. (9) we substitute

$$x=\frac{s(1+KZ)}{K} \quad ...(11)$$

then it reduces to

$$x\frac{d^2F}{dx^2}+n\frac{dF}{dx}-xF=0.$$

Taking

$$F=\left(\frac{x}{s}\right)^{\frac{1-n}{2}}f(x) \quad ...(12)$$

the above equation transforms into

$$x^2\frac{d^2f}{dx^2}+x\frac{df}{dx}-(m^2+x^2)f=0$$

where

$$m = \frac{1-n}{2}. \qquad \ldots(13)$$

The solution of which is

$$f = C_1 I_m(x) + D_1 K_m(x)$$

where C_1, D_1 are constants and I_m, K_m are modified Bessel functions of order m.

Since u_θ is finite as $Z \to \infty$

$$\therefore\ C_1 = 0 \text{ and } f = D_1 K_m(x). \qquad \ldots(14)$$

From (5), (6), (10), (12) and (14)

$$u_\theta = A\left(\frac{x}{s}\right)^m K_m(x)\, J_1(\lambda r)\, e^{ipt}$$

where

$$A = CD_1.$$

Again substituting the value of u_θ and μ in the first relation of (2) we get

$$\tau_{r\theta} = -A\mu_0 \lambda K^{-m}(1+KZ)^{1-m} J_2(\lambda r) K_m\left(\frac{s(1+KZ)}{K}\right) \times e^{ipt}.$$

In the case when the curved surface is free from stress the boundary condition is

$$\tau_{r\theta} = 0 \text{ on } r = a$$

which yields the frequency equation

$$J_2(\lambda a) = 0. \qquad \ldots(15)$$

The first ten positive non zero roots of the frequency equation (15) by Abramovitz and Stegun[9] are

5.13562, 8.41724, 11.61984, 14.79595, 17.95982, 21.11700, 24.27011, 27.42057, 30.56920, 33.71652.

The displacement and stress components are given by

$$u_\theta = AK^{-m}(1+KZ)^m K_m\left(\frac{s}{k}(1+KZ)\right) J_1(\lambda r)\, e^{ipt} \qquad \ldots(16)$$

$$\tau_{r\theta} = -A\mu_0 K^{-m}(1+KZ)^{1-m} \lambda K_m\left(\frac{s(1+KZ)}{K}\right) J_2(\lambda r)\, e^{ipt} \qquad \ldots(17)$$

$$\tau_{\theta Z} = -A\mu_0 s K^{-m}(1+KZ)^{1-m} K_{m-1}\left(\frac{s(1+KZ)}{K}\right) J_1(\lambda r)\, e^{ipt}. \qquad \ldots(18)$$

Next we consider the case when the curved surface of the cylinder is rigid. In this case the boundary condition is

$$\mu_\theta = 0 \text{ when } r = a$$

which in turn gives

$$J_1(\lambda a) = 0. \qquad \ldots(19)$$

The first ten non zero positive roots of the frequency equation (19) as given by Abramovitz and Stegun are

3.83171, 7.01559, 10,17347, 13.32369, 16.47063, 19.61586, 22.76008, 25.90367, 29.04683, 32.18968.

The roots in this case are, in order, smaller than those of the stress free case.

The expression for displacement and stresses in this case takes the form

$$u_0 = AK^{-m}(1 + KZ)^m J_1(\lambda r) K_m\left(\frac{s(1 + KZ)}{K}\right) e^{ipt} \qquad \ldots(20)$$

$$\tau_{r\theta} = -A\mu_0\lambda K^{-m}(1 + KZ)^{1-m} J_2(\lambda r) K_m\left(\frac{s(1 + KZ)}{K}\right) e^{ipt} \qquad \ldots(21)$$

$$\tau_{\theta Z} = -A\mu_0 sK^{-m}(1 + Z)^{1-m} K_{m-1}\left(\frac{s(1 + KZ)}{K}\right) J_1(\lambda r) e^{ipt}.$$

Particular Cases

(a) In order to obtain the solution when the material of the cylinder is homogeneous isotropic and the curved surface of the cylinder is stress free we substitute $n = 0$ i.e., $m = \frac{1}{2}$ in the equations (16) to (18) and they reduce to

$$u_\theta = A\sqrt{\frac{\pi}{2s}}\; e^{-s/K(1+KZ)} J_1(\lambda r) e^{ipt}$$

$$\tau_{r\theta} = -A\mu_0\lambda\sqrt{\frac{\pi}{2s}}\; J_2(\lambda r) e^{-s/K}(1 + KZ) e^{ipt}$$

$$\tau_{\theta Z} = -A\mu_0\sqrt{\frac{\pi s}{2}}\; J_1(\lambda r) e^{-s/K}(1 + KZ) e^{ipt}.$$

(b) When the Material of the cylinder is nonhomogeneous and the nonhomogeneity is characterised by the equation

$$\mu = \mu_0(1 + KZ)^2, \; \rho = \rho_0(1 + KZ)^2$$

and the curved surface of the cylinder is free from traction, the solution will be obtained by setting $n = 2$ i. e., $m = -\frac{1}{2}$ in equations (16) to (18). The above type of expressions for the modulus of rigidity and density are taken in view of the realistic. Earth model B of Bullen[10]. For instance, using the value $K = \cdot 0005$, $\mu_0 = 0.625$,

$\rho_0 = 3.22$ for a depth of 33 km we obtain the density and rigidity by using the above forms in close agreement with the data of Bullen at subsequent depth. Now for this case the displacement and stress components are

$$u_\theta = A\sqrt{\frac{\pi}{2s}}\left(\frac{K}{1+KZ}\right) J_1(\lambda r)\, e^{-s/K(1+KZ)}\, e^{ipt}$$

$$\tau_{r\theta} = -A\mu_0 K\lambda(1+KZ)\sqrt{\frac{\pi}{2s}}\, e^{-s/K(1+KZ)} J_2(\lambda r)\, e^{ipt}$$

$$\tau_{\theta Z} = -A\mu_0 K(K+s+KsZ)\, e^{-s/K(1+KZ)}\sqrt{\frac{\pi}{2s}}\, J_1(\lambda r)\, e^{ipt}.$$

(c) To obtain the displacement and stresses when the meterial of the cylinder is homogeneous and the curved surface of the cylinder is rigid we substitute $n = 0$, $m = \frac{1}{2}$ in eqns. (20 to 22) which in turn gives

$$u_\theta = A\sqrt{\frac{\pi}{2s}}\, J_1(\lambda r)\, e^{-s/K(1+KZ)}\, e^{ipt}$$

$$\tau_{r\theta} = -A\mu_0\lambda\sqrt{\frac{\pi}{2s}}\, J_2(\lambda r)\, e^{-s/K(1+KZ)}\, e^{ipt}$$

$$\tau_{\theta Z} = -A\mu_0\sqrt{\frac{\pi s}{2}}\, J_1(\lambda r)\, e^{-s/K(1+KZ)}\, e^{ipt}.$$

(d) When the material of the cylinder is non homogeneous, the nonhomogeneity is characterised by the equation

$$\mu = \mu_0(1+KZ)^2,\ \rho = \rho_0(1+KZ)^2$$

and the curved surface of the cylinder is rigid, we have to substitute $n = 2$, $m = -\frac{1}{2}$ in eqns. (20) to (22) and they take the form

$$u_\theta = A\sqrt{\frac{\pi}{2s}}\left(\frac{K}{1+KZ}\right)^{-s/K(1+KZ)} J_1(\lambda r)\, e^{ipt}$$

$$\tau_{r\theta} = -A\sqrt{\frac{\pi}{2s}}\,\mu_0 K(1+KZ)\lambda J_2(\lambda r)\, e^{-s/K(1+KZ)}\, e^{ipt}$$

$$\tau_{\theta Z} = -A\mu_0\sqrt{\frac{\pi}{2s}}\, K(K+s+KsZ) J_1(\lambda r)\, e^{-s/K(1+KZ)}\, e^{ipt}.$$

Numerical Results and Discussion

To obtain the displacement and stress for first mode of vibration, we take

$$\frac{s}{K} = 1,\ Ka = 1,\ \frac{Z}{a} = 1 \text{ and } K = .0005.$$

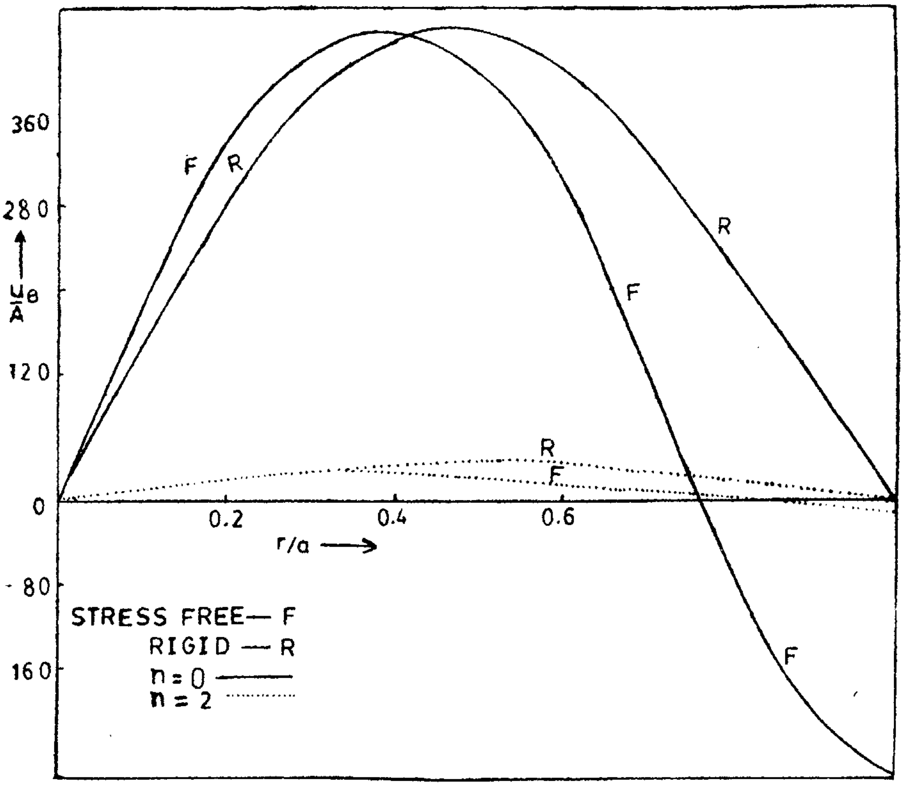
360
280
120
0
- 80
160
F
R
R
F
F
R
F
0.2
0.4
0.6
r/a
STRESS FREE— F
RIGID — R
n = 0
n = 2

FIG. 1.

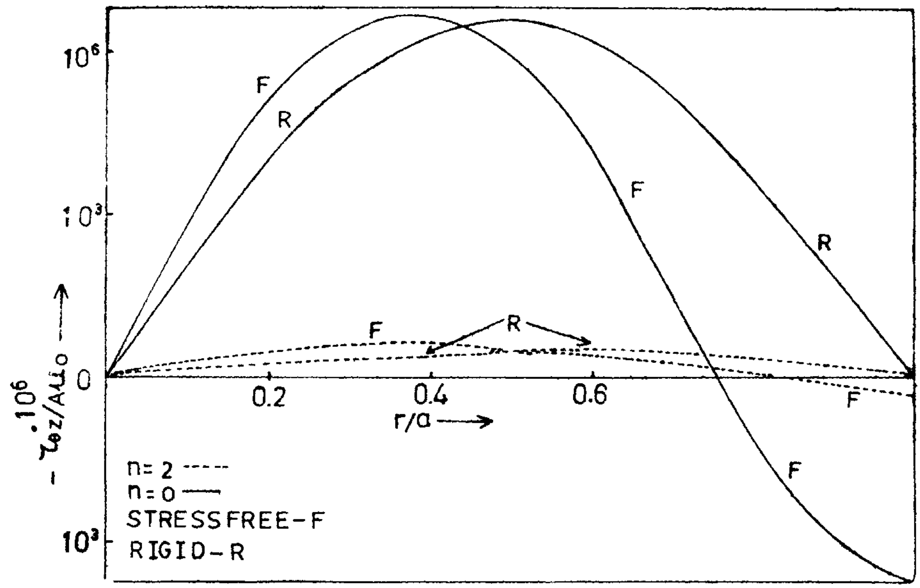
10^6
10^3
0
10^3
F
R
F
R
F
R
F
F
0.2
0.4
0.6
r/a
n= 2
n= 0
STRESSFREE-F
RIGID-R

FIG. 2.

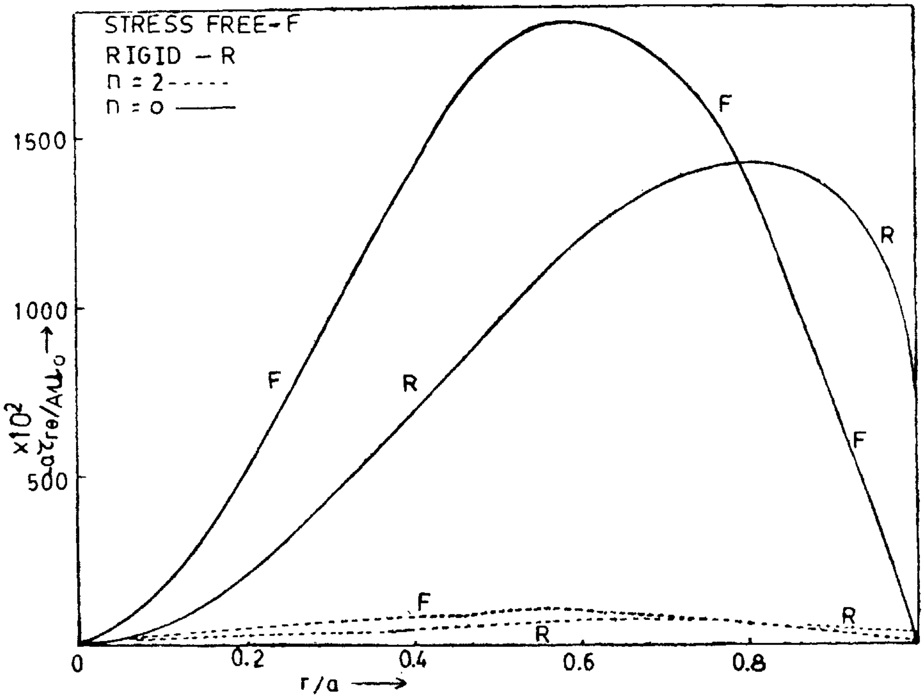

FIG. 3.

The distribution of $\frac{u_\theta}{A} \times 10^2$ and $\frac{u_\theta}{A} \times 10^4$ at different radii for $n = 0$, 2 are shown in Fig. 1 both for rigid and stress free boundary. The stress components, $-\frac{\tau_{\theta Z}}{A\mu_0}$ and $-\frac{a\tau_{r\theta}}{A\mu_0}$ are shown in Figure 2 and 3, respectively.

The roots of the frequency equation for stress free boundary are greater than those of the rigid boundary. For higher modes of vibration there exist different values of r for which u_θ vanishes. In all the cases the nonhomogeneity decreases the values of u_θ, $-\tau_{r\theta}$ and $-\tau_{\theta Z}$ at every point within the cylinder. The maximum value of the displacement and stresses occurs near $r/a = 0.5$ for rigid boundary and $r/a = 0.4$ for stress free boundary but it increases by an amount 0.2 for the case of $\tau_{r\theta}$. For the the case of stress free boundary the displacement and $\tau_{\theta Z}$ change their sign within the interval $0.7 < r/a < 0.8$.

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