Indian J. pure appl. Math., 20 (5) : 433–438, May 1989

A NOTE ON NORMED NEAR-ALGEBRAS

T. SRINIVAS AND K. YUGANDHAR

Department of Mathematics, Kakatiya University, Warangal 506009 (*A. P.*)

(*Received* 29 *March* 1988)

In this paper an attempt has been made to extend the theory of Algebras to Normed near-algebras. For this purpose we are led to a definition of a Normed near-algebra which differs from previous definition. A technique has been developed to find the regular elements in the Normed near-algebras and proved that the set of all regulas and quasi-regular elements is open. Further it has been shown that the concepts of topological divisors of zero and the spectrum of an element can be carried to Normed near-algebras also.

§1. Near-algebras were studied by Yamamuro[6], Brown[2] and others (cf. Pilz[3]). In order to extend the theory of Normed algebras to Normed near-algebras we have defined the Normed near-algebra with an additional condition.

Definition 1—A (right) near-algebra B over a field F is a linear space over F on which a multiplication is defined such that (i) B forms a semigroup under multiplication, (ii) multiplication is right distributive with respect to addition, (iii) $\alpha(xy) = (\alpha x)y$ for all $x, y \in B$ and $\alpha \in F$.

Definition 2— A near-algebra B over the real or complex numbers is called a Normed near-algebra provided that there is associated with each $x \in B$ a real number $\| x \|$, called the norm of x, with the following properties :

(i) $\| x \| \geqslant 0$ and $\| x \| = 0$ if and only if $x = 0$ (the additive identity of B),

(ii) $\| x + y \| \leqslant \| x \| + \| y \|$, for all $x, y \in B$,

(iii) $\| \alpha x \| = | \alpha | \, \| x \|$, α a real or complex number,

(iv) $\| xy \| \leqslant \| x \| \, \| y \|$, for all $x, y \in B$,

(v) $\| xy - xz \| \leqslant \| x \| \, \| y - z \|$, for all $x, y, z \in B$,

(vi) If B has an identity e, then $\| e \| = 1$.

Through out this paper the property (v) plays a crucial role in proving most of the theorems. It follows from the definition (2) that all the operations are continuous.

Through out this paper we assume that B is a normed near-algebra such that

(i) B is complete with respect to the norm defined as above,

(ii) B has an identity e,

(iii) For every $x \in B$, $0.x = x.0 = 0$.

Example— Let S be a Banach space over the field F of real or complex numbers. Then the set $B(S, S)$ of all Lipschitz continuous functions $x : S \to S$ such that $x(0) = 0$, and having the Lipschitz norm and with the composite map $x(y(s)) = (xy)(s)$ of S onto S, for all $x, y \in B(S, S)$ and $s \in S$, constitute a normed near-algebra[4,5].

Definition 3— An element $r \in B$ is said to be left (right) regular if there exists an element $s \in B$ such that $sr = e$ ($rs = e$).

The element s is called left (right) inverse for r. An element which is both left and right regular is called regular element. An element which is not (left, right) regular is called (left, right) singular.

Through out this we denote the set of all regular elements by G and the set of all singular elements by S.

The following Lemma is very useful in proving the main theorem.

Lemma 1— Let B be normed near-algebra, then every element $r \in B$ such that $\| e - r \| < 1$ is regular.

PROOF : Let $x = e - r$ and $y_0 = e$. Define the sequence $\{y_n\}_{n=1}^{\infty}$ inductively by

$$y_{n+1} = y_0 + xy_n.$$

Then

$$\begin{aligned} \| y_{n+1} - y_n \| &= \| xy_n - xy_{n-1} \| \\ &\leqslant \| x \| \, \| y_n - y_{n-1} \| \\ &\leqslant \| x \|^n \, \| y_1 - y_0 \| \\ &= \| x \|^{n+1} \end{aligned}$$

and hence $\| y_{n+p} - y_n \| \leqslant \| x \|^{n+p} + \ldots + \| x \|^{n+1}$ for any integer p.

Therefore $\{y_n\}$ is a Cauchy's sequence in B since $\| x \| < 1$. As B is complete, there exists a $y \in B$ such that $\lim_{n\to\infty} y_n = y$. From the relation $y_{n+1} = y_0 + xy_n$, we get $y = e + xy = e + (e - r)\, y$, that is $ry = e$.

Next consider

$$\begin{aligned} \| y_n r - e \| &= \| xy_{n-1} r - x \| \\ &\leqslant \| x \| \, \| y_{n-1} r - e \| \\ &\leqslant \| x \|^{n+1}. \end{aligned}$$

Therefore, $\| yr - e \| = \lim_{n\to\infty} \| y_n r - e \| = 0$, that is $yr = e$.

Hence r is regular.

Remark 1 : Here the inverse of r is given by $r^{-1} = \lim_{n\to\infty} y_n$, and it can be proved by using the same construction for y_n that

$$\| r^{-1} \| \leqslant \frac{1}{1 - \| e - r \|}.$$

Lemma 2— Let G_l denote the set of all left regular elements of B. If $x \in G_l$ then any element $z \in B$ satisfying $\| x - z \| < \| y \|^{-1}$ (where y is a left inverse of x) belongs to G_l.

PROOF : $\| e - yz \| = \| yx - yz \|$

$$\leqslant \| y \| \| x - z \|$$

$$< 1.$$

This implies by the Lemma (1) that $yz \in G$ and hence $z \in G_l$, which proves the Lemma.

Similarly we can state :

Lemma 3— Let G_l denote the set of all right regular elements of B. If $x \in G_l$ such that the right inverse of x is y then any element $z \in B$ satisfying $\|x-z\| < \|y\|^{-1}$, belongs to G_r.

Now we will prove the main theorem of this paper.

Theorem 1— The set of all regular elements G is an open set in B.

PROOF : Let $x \in G = G_l \cap G_r$.

Consider the open ball

$$S(x) = \{z \in B / \| z - x \| < 1/\| x^{-1} \|\}.$$

Then for any $z \in S(x)$, by Lemma (2) and Lemma (3), z is regular. Hence $S(x) \subset G$. Therefore G is open.

Remark 2 : Infact it can be proved that G_l and G_r are open and hence $G = G_l \cap G_r$ is open.

Theorem 2— Let B be the normed near-algebra. The mapping $r \to r^{-1}$ is a homeomorphism of G onto G.

PROOF : It is sufficient to show that the mapping $r \to r^{-1}$ is continuous.

Let $x, r \in G$ such that

$$\| x - r \| < 1/2 \| r^{-1} \|.$$

Then

$$\begin{aligned} \| e - xr^{-1} \| &= \| rr^{-1} - xr^{-1} \| \\ &\leqslant \| r - x \| \, \| r^{-1} \| \\ &< 1/2. \end{aligned}$$

Hence $xr^{-1} \in G$ and $\| (xr^{-1})^{-1} \| < 2$, by Remark 1.

Consequently,

$$\begin{aligned} \| x^{-1} \| &= \| r^{-1} rx^{-1} \| \\ &\leqslant \| r^{-1} \| \, \| rx^{-1} \| \\ &= \| r^{-1} \| \, \| (xr^{-1})^{-1} \| \\ &\leqslant 2 \| r^{-1} \|. \end{aligned}$$

Therefore

$$\begin{aligned} \| r^{-1} - x^{-1} \| &\leqslant \| x^{-1} xr^{-1} - x^{-1} rr^{-1} \| \\ &\leqslant \| x^{-1} \| \, \| x - r \| \, \| r^{-1} \| \\ &\leqslant 2 \| r^{-1} \|^2 \| x - r \|. \end{aligned}$$

This shows that the mapping $r \to r^{-1}$ is continuous. Thus we have $r \to r^{-1}$ is homeomorphism.

Remark 3 : The above theorem implies that the set of all regular elements G is a topological group.

The following theorem can be proved by using the same techniques as in algebras.

Theorem 3— Let B be a normed near-algebra and $\{r_n\}$ a sequence of left (right) regular elements of B which converges to an element $r \in B$. If s_n is a left (right) inverse for r_n and if $\{s_n\}$ is a bounded sequence, then r is also a left (right) regular.

PROOF : Let $\{r_n\}$ be a sequence such that $r_n \to r$ in B. Let $\{s_n\}$ be a bounded sequence that is $\sup_n \| s_n \| = M < \infty$ and s_n is a left inverse of r_n.

As $r_n \to r$ we get, for $\epsilon = 1/M > 0$ there exists a number $n_0 \geqslant 1$ such that

$$\| r_n - r \| < 1/(M + 1) \text{ for } n \geqslant n_0.$$

Now consider

$$\begin{aligned} \| e - s_{n_0} r \| &= \| s_{n_0} r_{n_0} - s_{n_0} r \| \\ &\leqslant \| s_{n_0} \| \, \| r_{n_0} - r \| \end{aligned}$$

(*equation continued on p.* 437)

$$\leqslant \| s_{n_0} \| / (M + 1)$$

$$< 1.$$

This implies that $s_{n_0} r \in G$, that is $r \in G_l$.

§2. *Definition* 4— Let B be a normed near-algebra. An element $r \in B$ is said to be quasi-regular if $e - r$ is left invertible. (cf. Beidlemen and Cox[1]).

Lemma 4— Let B be a normed near-algebra, then every element $x \in B$ such that $\| x \| < 1$ is quasi-regular.

PROOF : This is obvious, since $\| x \| < 1$ implies that $\| e - (e - x) \| < 1$. Hence by Lemma 1, $e - x$ is invertible, that is x is quasi-regular.

Theorem 4— Let Q denote the set of all quasi-regular elements of B. Then Q is open in B.

PROOF : We know that G_l is an open set in B. But an element $r \in Q \Leftrightarrow (e - r) \in G_l$. Hence Q is open in B.

Definition 5— An element z in the normed near-algebra B is called a left (right) topological divisor of zero provided there exists a sequence $\{z_n\}$ in B such that $\| z_n \| = 1$, for all n and $z z_n \to 0$ ($z_n z \to 0$).

We call the element which is either a left or right topological divisor of zero as a topological divisor of zero. We denote this set by Z and the set of all left (right) topological divisors of zero in B by Z_l (Z_r).

We now state the following theorems without proofs since they are very easy and can be proved using the same techniques as in algebras.

Theorem 5— The set of all topological divisors of zero is a subset of the set of all singular elements. In other words in the normed near-algebra B every topological divisor of zero is singular.

Theorem 6— The boundary of S in B is a subset of Z.

Theorem 7—(a) Every element z for which $(e - z)$ is a topological divisor of zero is quasi-singular, that is not quasi-regular.

(b) $G_l \subseteq H_l$, $G_r \subseteq H_r$ and $G \subseteq H$, where H_l, H_r and H are the sets that are complements of the sets G_l, G_r and G respectively.

(c) $G = G_l \cap H_r = G_r \cap H_l$.

(d) $G_l \cap S_r \subseteq Z_r$.

Definition 6—Let B a complex normed near-algebra and let x be any element of B. Let $\sigma_0(x)$, be the set of all non-zero complex numbers such that $e - x\lambda^{-1}e$ is singular. Put $\sigma(x) = \sigma_0(x)$, if x is not singular and $\sigma(x) = \sigma_0(x) \cup \{0\}$, if x is singular.

Then $\sigma(x)$ is defined as the spectrum of x.

The complement of $\sigma(x)$ is denoted by $\rho(x)$.

Theorem 8— $\sigma(x)$ is bounded and closed in B.

PROOF : Let $\lambda \in \sigma_0(x)$. By the Lemma 1, we get $\| e - (e - x\lambda^{-1} e) \| \nless 1$. That is $\| x\lambda^{-1} e \| \nless 1$, which implies that $\| x \| \geqslant | \lambda |$. Hence $\sigma(x)$ is bounded.

The mapping $\lambda \to x\lambda^{-1} e$ is continuous mapping from $\mathbb{C} - \{0\}$ to B. Hence Theorem 1, gives that $\sigma_0(x)$ is closed and hence $\sigma(x)$ is closed. Thus the proof is complete.

But it is an open problem to settle whether $\sigma(x)$ is non-empty or not when x is not singular.

ACKNOWLEDGEMENT

The authors are very much grateful to Professor A. Radhakrishna, Department of Mathematics, Kakatiya University, for his helpful conversations and valuable guidance.

REFERENCES

1. J. C. Beidleman and R. H. Cox, *Arch Math*. (Basel) **18** (1967). 485–92.
2. H. Brown, *Illinois J. Math*. **12** (1968), 215–27.
3. G. Pilz, *Near-rings*. North-Holland Math. Studies 23. New York, 1977.
4. V. Seppala, *Rc. Math. Applic*. **11** Ser. VI-(1978), 455-78.
5. V. Seppala, *Non Linear analysis, Theory, Methods and Applications*, **12** (1988), 185-208.
6. S. Yamamuro, *J. Aust. Math. Soc*. **7** (1967), 160–64.