Indian J. pure appl. Math., 20 (5) : 446-454, May 1989

SOME RESULTS ON STABILITY OF DIFFERENTIAL SYSTEMS WITH IMPULSIVE PERTURBATIONS

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(*Received* 22 *March* 1988)

The purpose of the present work is to derive a few results which improve upon some earlier finding on stability analysis of solutions of ordinary differential systems. Firstly, we treat a most general differential system with impulsive perturbations and study the stability of the solutions of such a system. The results thus derived unify and improve the results of Raghvendra and Rao[4] and Strauss and Yorke[5] simultaneously. Further we also treat the stability problem of a weakly linear differential system which is new and infact is an extension of result of Raghavendra and Rao. One final result is derived on necessary and sufficient condition for stability of specific nonlinear differential system.

1. INTRODUCTION

Raghavendra and Rao[4] investigated some stability properties of solutions of ordinary differential systems with respect to impulsive perturbation. In fact slight modification of Gronwall-Bellman type integral inequalities and their extentions, which has been given by them, is the main tool in considering the stability theorem for solutions of ordinary differential equations containing measures.

In the present work we try to deal with most general type of differential system analogous to Strauss and Yorke[5] with a difference that we now introduce impulsive perturbation in the system considered by him. Subsequently the stability theorem thus developed extends in theory as purported by Raghavendra and Rao[4]. The next section is devoted to preliminaries and basic lemmas quoted from Raghavendra and Rao[4] and Strauss and Yorke[5] which we have used later on in the sequel. Subsequent sections deal with the main results of the present work.

2. PRELIMINARIES AND BASIC LEMMAS

For any vector $x \in R^n$, the euclidean space of dimension n, let $| x | = \sum_{i=1}^{n} | x_i$. By $C[E, R^n]$ we denote the class of continuous mappings from E into R^n.

Let $J = [0, \infty]$. Most general type of differential system with impulsive perturbation is given as follows.

$$Dx = f(t, x) + g(t, x) Dv + h(t) \qquad ...(2.1)$$

where $x \in R^n$, Dv denotes the distributional derivative of the function $v, f, g \in C[J \times R^n, R^n]$ and $v : J \to R$ is a function of bounded variation and it is right continuous on J. Dv can also be identified with a Stieltjes measure and in fact possesses the effect of the sudden change of the state of the system at the points of discontinuity of u.

Equations (2.1) may be regarded as a perturbed system of ordinary differential equation

$$Dx = f(t, x) \qquad ...(2.2)$$

where the perturbation $g(t, x) Dv$ is impulsive and $h(t)$ satisfies certain smoothness condition of the type as discussed in Strauss and Yorke[5], to be stated later on in the sequel. We also assume that $g(t, x)$, $f(t, x)$ and $h(t)$ satisfy certain smoothness conditions sufficient to guarantee the existence and the uniqueness of the solutions of (2.1).

We assume the following conditions :

(G_1) There exists $\alpha > 0$ such that if $|x| \leqslant \alpha$, then $|g(t, x)| \leqslant \gamma(t)$ for all $t \geqslant 0$, where

$$G(t) = \int_t^{t+1} \gamma(s)\, dv(s) \to 0 \text{ as } t \to \infty \qquad ...(2.3)$$

where $\gamma \in C[J, R_+]$.

(G_2) There exists a continuous, non-increasing function $H(t)$ satisfying :

$$\lim_{t \to \infty} H(t) = 0$$

such that $\left| \int_{t_0}^{t} h(s)\, ds \right| \leqslant H(t_0)$ for every

$0 \leqslant t_0 \leqslant t \leqslant t_0 + 1$.

We shall prove that if $g(t, x)$ satisfies (G_1) and $h(t)$ satisfies (G_2) then there exists $T_0 \geqslant 0$ and $\delta_0 > 0$ such that if $t_0 \geqslant T_0$ and $|x_0| < \delta$, the solution $F(t, t_0, x_0)$ of (2.1) approaches zero as $t \to \infty$. In particular if $x = 0$ is a solution of (2.1) then it is uniformly asymptotically stable. In fact on the other hand if $g(t, x) = 0$ and $h(t)$ does not satisfy (G_2), then no solution of (2.2) tends to zero as $t \to \infty$.

In the case $f(t, x) = Ax$, where all the characteristic roots of A have negative real parts, we derive some stability results for the so called weakly nonlinear differential system of the form

$$Dx = Ax + F(t, x) + g(t, x)\, Dv + h(t) \qquad ...(2.4)$$

we now give some definition in the next paragraph.

Definition 2.1—The null solution of (2.1) is said to be uniformly asymptotically stable if the following two conditions hold :

(i) for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ and
$\tau = \tau(\epsilon) > 0$ such that $| y(t, t_0, x_0) | < \epsilon$,
$t \geqslant t_0 \geqslant \tau(\epsilon)$ provided that $| x_0 | < \delta$.

(ii) for every $\eta > 0$, there exists positive numbers
δ_0, τ_0 and $T = T(\eta)$ such that $| y(t, t_0, x_0) | < \eta$
$t \geqslant t_0 + T$, $t_0 > \tau_0$ provided $| x_0 | < \tau_0$.

In the next section we would extend the proof of the Theorem 3.1 of Raghavendra and Rao[4] to get our result and the proof thus given does not make use of the Lyapunov function as also done in the case of Theorem 5.1 of Strauss and Yorke[5]. Before we prove our result we give some lemmas which are basically the extensions of the lemmas which appear in Strauss and Yorke[5] and we also state the Gronwall type of inequalities quoted from Strauss and Yorke[5] which has been used further in our result. We also remark here that the generalized inequality of Raghavendra and Rao[4] is not needed in the proof of our result, instead the more traditional inequality such as the one given in Lemma 2.4 below suffices for the derivation of Theorem 3.1.

Lemma 2.1— $\int_{t_0}^{t} \gamma(s)\, dv(s) \leqslant \int_{t_0-1}^{t} G(s)\, ds$ for all $t > t_0 \geqslant 1$.

Lemma 2.2— $\int_{t_0}^{t} e^{\sigma s} \gamma(s)\, dv(s) \leqslant \int_{t_0-1}^{t} e^{\sigma(s+1)} G(s)\, ds$ fo all $\sigma > 0$ $t \geqslant t_0 \geqslant 1$.

Lemma 2.3— $\lim_{t \to \infty} e^{-\sigma t} \int_{1}^{t} e^{\sigma s} \gamma(s)\, dv(s) = 0$ for all $\sigma > 0$.

Lemma 2.4—Let $\gamma(t)$ and $p(t)$ be non-negative and continuous for $t \geqslant t_0$, let $C \geqslant 0$, $k \geqslant 0$, and let

$$\gamma(t) \leqslant C + \int_{t_0}^{t} [k\,\gamma(s) + p(s)]\, ds$$

then

$$\gamma(t) \leqslant C\, e^{k(t-t_0)} + \int_{t_0}^{t} p(s)\, e^{k(t-s)}\, ds.$$

The following theorem gives the sufficient conditions for uniform asymptotic stability of (2.1) with respect to (2.2). Let $f(t, 0) = 0$ for all $t \geqslant 0$.

Theorem 3.1—Let the null solution of (2.2) be uniformly asymptotically stable. Assume that :

(i) f satisfies the uniform Lipschitz continuous condition of the following type.

$$|f(t, x) - f(t, y)| \leqslant L|x - y|, \quad |x|, \quad |y| \leqslant a$$

where a is a positive real number.

(ii) $g(t, x)$ satisfies the condition as in (G_1) and (2.3) holds.

Then there exists $T_0 \geqslant 0$ and $\delta_0 > 0$ such that $t_0 \geqslant T_0$ and $|x_0| < \delta_0$, the solution $y(t, t_0, x_0)$ of (2.1) satisfies $|y(t, t_0, x_0)| \to 0$ as $t \to \infty$. In particular if $g(t, 0) = 0$ then the null solution of (2.1) is eventually asyptotically stable.

Proof : The solutions and constants corresponding to system (2.2) shall be starred, those for (2.1) shall not. Now let

$$Q(t) = \text{Sup}\{G(s) : t - 1 \leqslant s < \infty\}.$$

Then it is easy to see that

$$Q(t) \downarrow 0 \text{ as } t \to \infty.$$

Invoking lemma 2.1 we get

$$\int_{t_0}^{t} \gamma(s)\, dv(s) \leqslant \int_{t_0-1}^{t} G(s)\, ds \leqslant Q(t_0)(t - t_0 + 1)$$

if $t \geqslant t_0 \geqslant 1$. Also we have from condition (G_2)

$$\left|\int_{t_0}^{t} h(s)\, ds\right| \leqslant \left|\int_{t_0}^{t_0+1} h(s)\, ds\right| + \ldots\ldots + \left|\int_{t_0+m}^{t} h(s)\, ds\right|$$

$$\leqslant H(t_0) \ldots\ldots\ldots + H(t_0 + m)$$

$$\leqslant H(t_0)(t - t_0 + 1).$$

Put $B(t) = Q(t) + H(t)$. Let $t_0 \geqslant 1$ and $|x_0| \leqslant \gamma$.

If $y(t, t_0, x_0)$ is a solution of (2.1) and if

$|y(t, t_0, x_0)| \leqslant \gamma$ on $[t_0, t_0 + \tau]$ for some $\gamma > 0$, then

$|y(t, t_0, x_0) - y^*(t, t_0, x_0)|$

$$= \left| x_0 + \int_{t_0}^{t} f(s, y(s, t_0, x_0))\, ds + \int_{t_0}^{t} g(s, y(s, t_0, x_0))\, dv(s)\right.$$

(*equation continued on p.* 450)

$$+ \int_{t_0}^{t} h(s)\,ds - x_0 - \int_{t_0}^{t} f(s, y^*(s, t_0, x_0))\,ds \,|$$

$$\leqslant \int_{t_0}^{t} L\,|\,y(s) - y^*(s)\,|\,ds + \int_{t_0}^{t} \gamma(s)\,dv(s) + \left| \int_{t_0}^{t} h(s)\,ds \right|$$

$$\leqslant \int_{t_0}^{t} L\,|\,y(s) - y^*(s)\,|\,ds + B(t_0)(t - t_0 + 1).$$

From Lemma 2.4,

$$|\,y(t, t_0, x_0) - y^*(t, t_0, x_0)\,|$$

$$\leqslant B(t_0)\,e^{L(t-t_0)} + \int_{t_0}^{t} B(t_0)\,e^{L(t-s)}\,ds$$

$$\leqslant B(t_0)\,e^{L\tau} + B(t_0)\,e^{L\tau}(t - t_0)$$

$$\leqslant e^{L\tau}(1 + \tau)\,B(t_0).$$

We may assume without loss of generality that $\gamma \leqslant \delta_0^*$. Next we proceed to show by the estimates as derived earlier[4,5] that $|\,y(t, t_0, x_0)\,| < \epsilon$ on every interval $[t_0 + m\tau, t_0 + (m + 1)\tau]$ and hence on $[t_0, \infty)$. Hence if $g(t, 0) = 0$ and $h(t) = 0$, then it is clear that $x = 0$ is uniformly asymptotically stable. Rest of the proof is completed following Raghavendra and Rao[4]. In fact, given $0 < \eta < \gamma$ and choosing $\delta(\eta) = \delta^*(\eta/2)$, $0 < \delta < \eta$,

$\tau(\eta) = T^*(\delta/2)$ and $T_1(\eta)$ so that

$$B(T_1) < \delta\,[e^{L}(1 + \tau)\,L]^{-1}$$

we finally show that $|\,y(t, t_0, x_0)\,| < \eta$, for all $t > t_0 + T$,

where $T = T(\eta) = \tau + T_1$.

4. Weakly Non-linear Differential System

We consider a weakly linear differential system with impulsive perturbation in the present section. The results on stability derived subsequently generalizes Theorem 4.1 of Raghavendra and Rao[4]. We consider the following ordinary differential system.

$$Dx = Ax + F(t, x) + g(t, x)\,Dv + h(t) \qquad \text{...(4.1)}$$

where A is $n \times n$ constant matrix and $F \in C[J \times R^n, R^n]$. The solution $y(t)$ of (4.1) satisfying $y(t_0) = x_0$, $t_0 \in J$, is given by

$$y(t) = \Phi(t - t_0)\,x_0 + \int_{t_0}^{t} \Phi(t - s)\,F(s, y(s))\,ds$$

(*equation continued on p.* 451)

$$+ \int_{t_0}^{t} \Phi(t-s)\, g(s, y(s))\, dv(s)$$

$$+ \int_{t_0}^{t} \Phi(t-s)\, h(s)\, ds \qquad \ldots(4.2)$$

where $\Phi(t)$ is the fundamental matrix of the equation $x' = Ax$ satisfying $\Phi(0) = I$ and $\Phi \in C^{\infty}(J)$ where I denotes the unit matrix[4].

Theorem 4.1—Suppose that

(i) all characteristic roots of A have negative real parts

(ii) given $\epsilon > 0$, there exists $\delta(\epsilon)$, $T(\epsilon) > 0$ such that

$$|F(t, x)| \leqslant \epsilon\, |x| \text{ provided } |x| < \delta(\epsilon) \text{ and } t \geqslant T(\epsilon)$$

(iii) The condition (G_1) and (G_2) of section 2 hold.

Then, there exists $T_0 \geqslant 0$ and $\delta_0 > 0$ such that for every $t_0 \geqslant T_0$ and $|x_0| < \delta_0$, any solution $y(t) = y(t, t_0, x_0)$ of (4.1) satisfies

$$\lim_{t \to \infty} y(t) = 0.$$

If, in particular (4.1) possesses null solution then the null solution is uniformly asymptotically stable.

Proof : We give a brief outline of the proof. Also note that the technique is same as in Raghavendra and Rao[4]. Because all characteristic roots of A have negative real parts, there exists positive constants a and c_1, such that

$$|\Phi(t)| = |e^{At}| \leqslant c_1\, e^{at} \text{ for } t \geqslant 0. \qquad \ldots(4.3)$$

Assume that

$$\delta = \delta((a/2)\, c_1^{-1}) < \gamma$$

choose t_0 and x_0, so that $t_0 \geqslant T = T((a/2)\, C_1^{-1})$ and $|x_0| < \delta_0 < \delta$. So far as $|y(t)| = |y(t, t_0, x_0)| \leqslant \delta$, for $t > t_0$ we have

$$|y(t)| \leqslant c_1\, e^{-a(t-t_0)}\, |x_0| + \int_{t_0}^{t} (a/2)\, e^{-a(t-s)}\, |y(s)|\, ds$$

$$+ \int_{t_0}^{t} c_1\, e^{-a(t-s)}\, \gamma(s)\, dv(s) + \left| \int_{t_0}^{t} e^{-a(t-s)}\, h(s)\, ds \right|$$

which implies that

$$|y(t)|\, e^{at} \leqslant c_1\, e^{at_0}\, |x_0| + a/2 \int_{t_0}^{t} e^{as}\, |y(s)|\, ds$$

(*equation continued on p.* 452)

$$+ \int_{t_0}^{t} c_1 e^{as} \gamma(s) \, dv(s) + \int_{t_0}^{t} e^{as} h(s) \mid ds. \qquad \ldots(4.4)$$

Applying Lemma 2.4, we have from (4.4)

$$| y(t) e^{at} < c_1 e^{at_0} | x_0 | e^{a/2(t-t_0)} + \int_{t_0}^{t} c_1 e^{as} e^{a/2(t-s)} \gamma(s) \, dv(s)$$

$$+ \int_{t_0}^{t} e^{a/2(t-s)} e^{as} | h(s) | \, ds.$$

From which it follows that

$$| y(t) | \leqslant c_1 e^{-a/2(t-t_0)} | x_0 | + c_1 \int_{t_0}^{t} e^{-a/2(t-s)} \gamma(s) \, dv(s)$$

$$+ \int_{t_0}^{t} e^{-a/2(t-s)} | h(s) | \, ds.$$

By choosing δ_0 and T_0 suitably and using the properties of the function $Q(t)$ and $H(t)$ we can proceed as Raghavendra and Rao[4] to show that for $t \geqslant T_0$ and $| x_0 | \leqslant \delta$, we have $| y(t) | < \delta$, which implies the existence of $y(t, t_0, x_0)$ on the interval $[t_0, \infty)$. For $t > T_0 > 1$, the fact that

$$\int_{t_0}^{t} e^{-a/2(t-s)} \gamma(s) \, dv(s) \to 0 \quad \text{as } t \to \infty$$

follows by using the same arguments as in Raghavendra and Rao[4] by using property (G_1). We next show that

$$\int_{t_0}^{t} e^{-a/2(t-s)} | h(s) | \, ds \to 0 \quad \text{as } t \to \infty$$

we have

$$\int_{t_0}^{t} e^{-a/2(t-s)} | h(s) | \, ds = \int_{t_0}^{t/2} e^{-a/2(t-s)} | h(s) | \, ds$$

$$+ \int_{t_0}^{t} e^{-a/2(t-s)} | h(s) | \, ds$$

$$\leqslant | h(0) | e^{-a/2t} \int_{0}^{t/2} e^{-a/2s} \, ds + h(t/2) | \int_{t/2}^{t} e^{-a/2(t-s)} | \, ds$$

from which it is easy to see that the right hand side of the above inequality goes to zero as $t \to \infty$.

Hence

$$| y(t, t_0, x_0) | \to 0 \quad \text{as } t \to \infty.$$

This also proves that if $g(t, 0) = 0$ and $h(t) = 0$, then the null solution of (4.1) is eventually uniformly stable, which completes the proof.

5. A New Result

In the present section we give a new result on necessary and sufficient conditions on stability of a non-linear differential system with impulsive perturbation.

Theorem 5.1—Let $x = 0$ be uniformly asymptotically stable. Let the condition (i) be satisfied for $f(t, x)$ as in Theorem 3.1. Also let $g(t, x)$ satisfy the condition (G_1). Then for the following system

$$Gx = f(t, x) + g(t, x)\, Dv + h(t) \qquad \ldots(5.1)$$

the conclusions of Theorem 3.1 hold if and only if $h(t)$ satisfies condition (G_2).

Proof : Let h satisfy condition (G_2), then the proof of the Theorem 3.1 establishes this one way conclusion.

Conversely, suppose h does not satisfy the condition (G_2). Then there exist $\eta > 0$ and sequences $\{t_n\}$ and $\{\theta_n\}$ with $0 \leqslant \theta_n \leqslant 1$ for every n and $t_n \to \infty$ as $h \to \infty$, such that

$$\left| \int_{t_n}^{t_n+\theta_n} h(t)\, dt \right| > \eta \qquad \ldots(5.2)$$

for every n. Suppose that there exist some $t_0 \geqslant 0$ and some x_0 for which the solution $y(t, t_0, x_0)$ of (5.1) satisfies $| y(t, t_0, x_0) | \to 0$ as $t \to \infty$. Choose T so large that $t \geqslant T$ implies

$$| y(t, t_0, x_0) | < \eta\, [L]^{-1}$$

where L is the Lipschitz constant as in condition (I) of Theorem 3.1, we may assume without loss of generality that $L \geqslant 1$. Also from condition (G_1) it follows, choosing n large enough that we can get

$$\int_{t_n}^{t_n+\theta_n} \gamma(t)\, dv(t) < \frac{\eta}{4}$$

choose n so large that $t_n \geqslant T$, then we have

$$\left| \int_{t_n}^{t_n+\theta_n} h(t)\, dt \right| = \int_{t_n}^{t_n+\theta_n} f(t, y(t, t_0, x_0))\, dt$$

(*equation continued on p.* 454)

$$+ \int_{t_n}^{t_n+\theta_n} g(t, y(t, t_0, x_0))\, dv(t)$$

$$- \int_{t_n}^{t_n+\theta_n} y'(t, t_0, x_0)\, dt \,|$$

$$< \int_{t_n}^{t_n+\theta_n} L\, |\, y(t, t_0, x_0)\, |\, dt + \int_{t_n}^{t_n+\theta_n} \gamma(t)\, dv(t)$$

$$+ |\, y(t_n + \theta_n, t_n, x_0)\, | + |\, y(t_n, t_0, x_0)\, |$$

$$\leqslant \eta/4 + \eta/4 + \eta/4 + \eta/4 = \eta$$

which contradicts condition (5.2) hence $|\, y(t, t_0, x_0)\, |$ can not tend to zero as $t \to \infty$. Completing the proof of the theorem.

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