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REMARKS ON SUBMANIFOLDS OF CODIMENSION 2 OF AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

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We determine submanifolds M of codimension 2 in an even-dimensional Euclidean space E in relation to the integrability of the almost complex structure J on $M \times R^1 \times R^1$.

INTRODUCTION

Blair *et al.*[1] introduced the so-called (f, g, u, v, λ)-structure which is naturally defined in a submanifold of codimension 2 of an almost complex manifold. An even-dimensional sphere of codimension 2 of an even-dimensional Euclidean space is a typical example of a manifold admitting the structure. Okumura[2], Yano and Okumura[4] studied submanifolds admitting a normal (f, g, u, v, λ)-structure.

In the present paper, we shall study submanifolds of codimension 2 in an even-dimensional Euclidean space in relation to the Nijenhuis tensors formed by the induced (f, g, u, v, λ)-structure on the submanifolds.

1. SUBMANIFOLDS OF CODIMENSION 2 OF AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

Let E be a $(2n + 2)$-dimensional Euclidean space and X the position vector of a point of E from the origin of E. Since E is even-dimensional, it is regarded as a flat Kaehlerian manifold, that is, there exists a (1, 1)-tensor field F satisfying

$$F^2 = -I, \quad FY \cdot FZ = Y \cdot Z \qquad \ldots(1.1)$$

for any vectors Y and Z and

$$\tilde{\nabla} F = 0 \qquad \ldots(1.2)$$

where I denotes the identity transformation, a dot the inner product in E and $\tilde{\nabla}$ the Riemannian connection of E.

Let M be a $2n$-dimensional submanifold of E covered by a system of local coordinate neighbourhoods $\{U ; x^h\}$, where and in the sequel the indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$.

By putting $X_i = \partial_i X$ $(\partial_i = \partial/\partial x^i)$, X_i are $2n$ linearly independent local vector fields tangent to M and the induced Riemannian metric of M is given by $g_{ji} = X_j . X_i$.

We denote by C and D two mutually orthogonal unit normals to M such that X_i, C and D form the positive orientation of E.

The transforms FX_i, FC and FD of X_i, C and D by F can be expressed as

$$FX_i = f_i^{\ h} X_h, + u_i C + v_i D \qquad ...(1.3)$$

$$FC = - u^h X_h + \lambda D$$

$$FD = - v^h X_h - \lambda C \qquad ...(1.4)$$

where $f_i^{\ h}$ are components of a tensor field of type (1, 1), u_i and v_i 1-forms, λ a function of M, $u^h = u_i g^{ih}$ and $v^h = v_i g^{ih}$.

Applying F to (1.3), (1.4) and (1.5) and taking account of (1.1) and (1.3)$\sim$ (1.5) we have[2,4].

$$f_i^{\ t} f_t^{\ h} = - \delta_i^h + u_i u^h + v_i v^h \qquad ...(1.6)$$

$$u_t f_i^{\ t} = \lambda v_i, \ v_t f_i^{\ t} = - \lambda u_i \qquad ...(1.7)$$

$$u_i u^i = 1 - \lambda^2 = v_i v^i, \ u_i v^i = 0 \qquad ...(1.8)$$

$$f_j^{\ k} f_i^{\ h} g_{kh} = g_{ji} - u_j u_i - v_j v_i \qquad ...(1.9)$$

and $f_{ji} = f_j^{\ h} g_{hi}$ is skew symmetric in i and j

The totality (f, g, u, v, λ) satisfying eqns. (1.6) $\sim$ (1.9) is called an (f, g, u, v, λ)-structure[1].

The equations of Gauss and Weingarten are given by

$$\nabla_j X_i = h_{ji} C + k_{ji} D \qquad ...(1.10)$$

$$\nabla_j C = - h_j^{\ h} X_h + l_j D \qquad ...(1.11)$$

$$\nabla_j D = - k_k^{\ h} X_h - l_j D \qquad ...(1.12)$$

where h_{ji} and k_{ji} are the second fundamental tensors of M with respect to the normals C and D respectively, and l_j the third fundamental tensor of M in E and ∇ the Riemannian connection of M.

Differentiating eqns. (1.3) ~ (1.5) and taking account of (1.2), (1.10), (1.11) and (1.12), we have[2,4].

$$\nabla_j f_i^{\ h} = -h_{ji} u^h + h_j^{\ h} u_i - k_{ji} v^h + k_j^{\ h} v_i \qquad ...(1.13)$$

$$\nabla_j u_i = -h_{jk} f_i^{\ k} - \lambda k_{ji} + l_j v_i \qquad ...(1.14)$$

$$\nabla_j v_i = -k_{jk} f_i^{\ k} + \lambda h_{ji} - l_j u_i \qquad ...(1.15)$$

$$\nabla_j \lambda = -h_j^{\ k} v_k + k_j^{\ k} u_k. \qquad ...(1.16)$$

2. Nijenhuis Tensors

We consider the product manifold $M \times R^1 \times R^1$, R^1 being a 1-dimensional Euclidean space, and we denote it by $\bar{M}$. The indices $A, B, C, \ldots$ will run over the ranges $1, 2, \ldots, 2n, 2n+1, 2n+2$, and $*$ stands for $2n+1$ and $\#$ for $2n+2$. If we define on $\bar{M}$ a tensor field J of type (1,1) by local components

$$(J_B^{\ A}) = \begin{pmatrix} f_i^{\ j} & u_i & v_i \\ -u^j & 0 & \lambda \\ -v^j & -\lambda & 0 \end{pmatrix} \qquad ..(2.1)$$

in each $\{U \times R^1 \times R^1, x^A\}$, then we can see that $J^2 = -I$ on $\bar{M}$, that is, $\{\bar{M}, J\}$ becomes an almost complex manifold. The components of Nijenhuis tensors of the almost complex structure J are given by

$$N_{ji}^{\ h} = f_j^{\ k} \partial_k f_i^{\ h} - f_i^{\ k} \partial_k f_j^{\ h} - (\partial_j f_i^{\ k} - \partial_i f_j^{\ k}) f_k^{\ h} + (\partial_j u_i - \partial_i u_j) u^h + (\partial_j v_i - \partial_i v_j) v^h \qquad ...(2.2)$$

$$N_{ji}^{\ *} = f_j^{\ k} \partial_k u_i - f_i^{\ k} \partial_k u_j - (\partial_j f_i^{\ k} - \partial_i f_j^{\ k}) u_k + \lambda (\partial_j v_i - \partial_i v_j) \qquad ..(2.3)$$

$$N_{ji}^{\ \#} = f_j^{\ k} \partial_k v_i - f_i^{\ k} \partial_k v_j - (\partial_j f_i - \partial_i f_j) v_k - \lambda (\partial_j u_i - \partial_i u_j) \qquad ...(2.4)$$

$$N_{j*}^{\ h} = -f_j^{\ k} \partial_k v^h - v^k \partial_k f_j^{\ h} + (\partial_j u^k) f_k^{\ h} + (\partial_i \lambda) v^h \qquad ...(2.5)$$

$$N_{j\#}^{\ h} = -f_j^{\ k} \partial_k v^h + v^k \partial_k f_j^{\ h} + (\partial_j v^k) f_k^{\ h} - (\partial_j \lambda) v^h$$

$$N_{\#*}^{\ h} = v^k \partial_k u^h - u^k \partial_k v^h \qquad ...(2.7)$$

$$N_{*i}^{\#} = - u^k \partial_k v_i - f_i^k \partial_k \lambda - (\partial_i u^k) v_k \qquad \text{...(2.8)}$$

$$N_{*i}^{*} = - u^k \partial_k u_i - (\partial_i u^k) u_k - \lambda (\partial_i \lambda) \qquad \text{...(2.9)}$$

$$N_{j\#}^{*} = - f_j^k (\partial_k \lambda) - v^k (\partial_k u_j) + (\partial_j v^k) u_k \qquad \text{...(2.10)}$$

$$N_{j\#}^{\#} = v^k (\partial_k v_j) + (\partial_j v^k) v_k + \lambda (\partial_j \lambda) \qquad \text{...(2.11)}$$

$$N_{\#*}^{*} = - u^k (\partial_k \lambda) \qquad \text{...(2.12)}$$

$$N_{\#*}^{\#} = - v^k (\partial_k \lambda). \qquad \text{...(2.13)}$$

The Nijenhuis tensor of an almost complex structure J satisfies the identity[3]

$$N_{CE}^{A} J_B^E + N_{CB}^{E} J_E^A = 0.$$

Hence, substituting (2.1) into (2.4) and taking account of the algebraic properties of (f, g, u, v, λ)-structure, we can obtain the

Proposition 2.1—Let there be given an (f, g, u, v, λ) structure on M and let the function $\lambda (1 - \lambda^2)$ does not vanish almost everywhere. If N_{ji}^{h}, N_{ji}^{*} and $N_{ji}^{\#}$ vanish on M, so do the other components of the Nijenhuis tensors of J.

3. Submanifolds with Vanishing Nijenhuis Tensors

Let M be a submanifold of codimension 2 in a $(2n + 2)$-dimensional Euclidean space E. Then M admits an (f, g, u, v, λ)-structure and the equations of Gauss. Codazzi and Ricci on M are given by

$$K_{kji}^{h} = h_k^h h_{ji} - h_j^h h_{ki} + k_k^h k_{ji} - k_k^h k_{ki} \qquad \text{...(3.1)}$$

$$\nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} = 0 \qquad \text{...(3.2)}$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} = 0 \qquad \text{...(3.3)}$$

$$\nabla_j l_i - \nabla_i l_j + h_j^t k_{ti} - h_j^t k_{ti} - h_i^t k_{tj} = 0 \qquad \text{...(3.4)}$$

respectively.

Assume that the function $\lambda (1 - \lambda^2)$ does not vanish almost everywhere on M. In the case where theconnection induced in the bundle of M has null curvature, we say that the normal connection is trivial.

The following lemma is well known[4].

Lemma 3.1—If the tensor N^h_{ji} vanishes on M and the connection induced in the bundle is trivial, then there exist two constants α and β such that

$$h^t_j \; u^j = \alpha u^t, \; h^t_j \; v^j = \alpha v^t \qquad \ldots(3.5)$$

$$k^t_j \; u^j = \beta u^t, \; k^t_j \; v^j = \beta \, v^t \qquad \ldots(3.6)$$

$$h^h_j \; h_{th} = \alpha \, h_{jt}, \quad k^h_j \; k_{th} = \beta k_{jt} \qquad \ldots(3.7)$$

and h^h_i and k^h_i commute with f^h_i.

On the other hand, the tensor fields (2.2) $\sim$ (2.4) can be rewritten as

$$\begin{aligned} N^h_{ji} \; & (f^t_j \; h^h_t - h^t_j \; f^h_t) \, u_i - (f^t_i \; h^h_t \; h^h_t - h^t_i \; f^h_t) \, u_j \\ & + (f^t_j \; k^h_t - k^t_j f^h_t) \, v_i - (f^t_i \; k^h_t - k^t_i \; f^h_t) \, v_j \\ & + (l_j \, v_i - l_i \, v_j) \, u^h - (l_j \, u_i - l_i \, u_j) \, v^h. \end{aligned} \qquad \ldots(3.8)$$

$$\begin{aligned} N^*_{ji} = \; & l_k \, (f^k_j \; v_i - f^k_i \; v_j) + \lambda \, (l_i \, u_i - l_j \, u_i) \\ & - (h^k_j \; u_i + k^k_j \; v_i - h^k_i \; u_j - k^k_i \; v_j) \, u_k \end{aligned} \qquad (3.9)$$

$$\begin{aligned} N^{\#}_{ji} = \; & l_k \; (f^k_i \; u_j - f^k_j \; u_i) - \lambda \, (l_j \, v_i - l_i \, v_j) \\ & - v_k \; (h^k_j \; u_i + k^k_j \; v_i - h^k_i \; u_j - k^k_i \; v_j). \end{aligned} \qquad \ldots(3.10)$$

If $N^*_{ji} = 0$ in addition to the assumptions of Lemma 3.1, then we have

$$\beta \, u_j \, v_i - \beta \, u_i \, v_j = 0. \qquad \ldots(3.11)$$

Similarly, $N^{\#}_{ji} = 0$ implies

$$\alpha \, v_j \, u_i - \alpha \, v_i \, u_j = 0. \qquad \ldots(3.12)$$

Lemma 3.2—If the tensor N^h_{ji} vanishes on M and the connection induced in the normal bundle is trivial, then the following condition are equivalent to one another :

(1) The components N^*_{ji} vanishes.

(2) $\beta = 0$.

(3) $k_{ji} = 0$.

In this case, the equations of Gauss and Weingarten reduce to

$$\nabla_j X_i = h_{ji} C, \quad \nabla_j C = - h_j^h X_h, \quad \nabla_j D = 0$$

respectively, and consequently D is a constant vector and we have

$$\nabla_j (X \cdot D) = 0$$

that is, $X \cdot D =$ constant. Thus we have

Theorem 3.3—Let M be a submanifold of condimension 2 of E^{2n+2} and suppose the connection induced in the normal bundle of M is trivial. If N_{ji}^h and N_{ji}^* vanish, then M is a hypersurface of a hyperplane E^{2n+1}.

We prepare the following :

Theorem A^4—Let M be a 2n-dimensional complete differentiable hypersurface in *a* $(2n + 1)$—dimensional euclidean space E'. If the component N_{ji}^h vanishes on M, then M is a product of a sphere and *a* plane.

Combining Theorems 3.3 and A, we obtain :

Theorem 3.4—Let M be a complete submanifold of codimensian 2 of E^{2n+2} such that the connection induced in the normal bundle of M is trivial. If N_{ji}^h and N_{ji}^* vanish, then M is a product of a sphere and a plane.

Assuming $N_{ji}^{\#} = 0$ instead of $N_{ji}^* = 0$, we can obtain the same conclusion as Theorem 3.4. Therefore if the components N_{ji}^h, N_{ji}^* and $N_{ji}^{\#}$ vanish identically and the connection induced in the normal bundle is trivial, then we get

$$\nabla_j X_i = 0, \quad \nabla_j C = 0, \quad \nabla_j D = 0$$

and

Theorem 3.5—Let M be a submanifold of codimension 2 of E^{2n+2} with trivial normal connection. If the Nijenhuis tensors N_{ji}^h, N_{ji}^* and $N_{ji}^{\#}$ vanish on M, then M is a plane of codimension 2.

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References

1. D. E. Blair, G. D. Ludden and K. Yano, *Kodai Math. Sem. Rep.* **22** (1970), 188–95.
2. M. Okumura, Differential Geom., in honor of K. Yano, Kinokuniya, Tokyo, 1972, 373–83.
3. K. Yano, *Differential Geometry on Complex and Almost Complex Spaces.* Pergamon Press, 1965.
4. K. Yano and M. Okumura, *Kodai Math Sem. Rep.* **23** (1971) 172–97.