Indian J. pure appl. Math., **20** (5) : 472–482, May 1989

ALMOST IRRESOLUTE FUNCTIONS

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(*Received* 9 *June* 1988)

A function $f: X \to Y$ is said to be almost irresolute[10] if for each $x \in X$ and each semi-neighbourhood V of $f(x)$, the semi-closure of $f^{-1}(V)$ is a semi-neighbourhood of x. In this paper, we obtain several characterizations of almost irresolute functions and investigate the relationship between such functions and some weak forms of irresolute functions. We also improve on some results established by Dube *et al.*[9].

1. INTRODUCTION

Crossley and Hildebrand[6] defined irresolute functions by utilizing semi-open sets due to Levine[12]. Recently, as weak forms of irresoluteness, weak irresoluteness[9], θ-irresoluteness[9], almost irresoluteness[21] and quasi irresoluteness[8] have been defined and investigated independently. However, it will turn out that these four weak forms of irresoluteness are equivalent. On the other hand, Dube *et al.*[10] have introduced the notion of almost irresolute functions which is independent of that of almost irresolute functions in the sense of Thakur and Paik[21].

The purpose of the present paper is to investigate almost irresolute functions in the sense of Dube *et al.*[10]. Note that, after section 3, "almost irresolute" always means "almost irresolute" in the sense of Dube *et al.* In section 2, we point out that weak irresoluteness, θ irresoluteness, almost irresoluteness in the sense of Thakur and Paik and quasi irresoluteness are all equivalent. In section 3, we present several characterizations of almost irresolute functions in the sense of Dube *et al.* In section 4, we investigate the relationship among semi-continuity, quasi irresoluteness and almost irresoluteness in the sense of Dube *et al.*[10]. In section 5, we introduce

* This research was supported by a grant from the C.N.A. (G.N.S.A.G.A.) and M.P.I. through "Fondi 40%".

and investigate semi preopen functions. The last section concerns with strongly semi-closed graphs and contains several improvements on results established by Dube *et al.*[9].

2. PRELIMINARIES

In this section, we will point out that almost irresoluteness[21], weak irresoluteness[9], θ-irresoluteness[9] and quasi irresoluteness[8] are all equivalent.

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset S of X is said to be semi-open[12] if there exists an open set U of X such that $U \subset S \subset \mathrm{Cl}(U)$, or equivalently if $S \subset \mathrm{Cl}(\mathrm{Int}(S))$, where $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$ denote the closure of S and the interior of S, respectively. A subset S is called a semi-neighbourhood[3] of a point x of X if there exists a semi-open set U such that $x \in U \subset S$. The complement of a semi-open set is called semi-closed. A semi-closed and semi-open set is said to be semi-clopen. For a subset S of X, the intersection of all semi-closed sets containing S is called the semi-closure of S (Crossley and Hildebrand[5]) and is denoted by $s\,\mathrm{Cl}(S)$. The semi-interior of S, denoted by $s\,\mathrm{Int}(S)$, is defined by the union of all semi-open sets contained in S. The family of all semi-open sets of X is denoted by $SO(X)$. For each $x \in X$, the family of all semi-open sets containing x is denoted by $SO(X, x)$. A subset S is said to be regular-open if $S = \mathrm{Int}(\mathrm{Cl}(S))$. A subset S is said to be regular semi-open[4] if there exists a regular open set U of X such that $U \subset S \subset \mathrm{Cl}(U)$.

The following two lemmas are due to Di Maio and Noiri[7].

Lemma 2.1—The following are equivalent for a subset A of a space X:

(a) A is regular semi-open.

(b) $A = s\,\mathrm{Int}(s\,\mathrm{Cl}(A))$.

(c) A is semi-clopen.

Lemma 2.2—If $A \in SO(X)$, then $s\,\mathrm{Cl}(A)$ is semi-clopen.

Definition 2.3— A function $f: (X, \tau) \to (Y, \sigma)$ is said to be irresolute[6] (resp. semi-continuous[12]) if $f^{-1}(V) \in SO(X, \tau)$ for every $V \in SO(Y, \sigma)$ (resp. $V \in \sigma$).

Definition 2.4— A function $f: X \to Y$ is said to be (a) almost irresolute[21] if $f^{-1}(V) \in SO(X)$ for every regular semi-open set V of Y; (b) weakly irresolute[9] (resp. θ-irresolute[9]) if for each $x \in X$ and each semi-neighbourhood V of $f(x)$, there exists a semi-neighbourhood U of x such that $f(U) \subset s\,\mathrm{Cl}(V)$ (resp. $f(s\,\mathrm{Cl}(U)) \subset s\,\mathrm{Cl}(V)$); (c) quasi irresolute[8] if for $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in SO(X, x)$ such that $f(U) \subset s\,\mathrm{Cl}(V)$.

Lemma 2.5[8]—The following are equivalent for a function $f: X \to Y$:

(a) f is quasi irresolute.

(b) For each $x \in X$ and each $V \in SO\ (Y, f(x))$, there exists $U \in SO\ (X, x)$ such that $f\ (s\ \text{Cl}\ (U)) \subset s\ \text{Cl}\ (V)$.

(c) $f^{-1}\ (V)$ is semi-clopen in X for every semi-clopen set V of Y.

(d) $f^{-1}\ (V) \subset s\ \text{Int}\ (f^{-1}\ (s\ \text{Cl}\ (V)))$ for every $V \in SO\ (Y)$.

(e) $s\ \text{Cl}\ (f^{-1}\ (V)) \subset f^{-1}\ (s\ \text{Cl}\ (V))$ for every $V \in SO\ (Y)$.

Theorem 2.6— The following are equivalent for a function $f: X \to Y$:

(a) f is quasi irresolute.

(b) f is weakly irresolute.

(c) f is θ-irresolute.

(*d*) f is almost irresolute.

PROOF : This follows from Definition 2.4, Lemmas 2.1 and 2.5.

Remark 2.7 : It is shown in Di Maio and Noiri[8] that semi-continuity and quasi irresoluteness are independent of each other and they are implied by irresoluteness.

3. CHARACTERIZATIONS

In this section, we obtain several characterizations of almost irresolute functions in the sense of Dube *et al.*[10]. A subset S of a space X is said to be preopen[15] if $S \subset \text{Int}\ (\text{Cl}\ (S))$.

Definition 3.1— A subset S of a space X is said to be semi-preopen[1] if there exists a preopen set U in X such that $U \subset S \subset \text{Cl}\ (U)$.

The family of all semi-preopen sets in X is denoted by $SPO\ (X)$. The complement of a semi-preopen set is called semi-preclosed[1].

Lemma 3.2[1]—The following are equivalent for a subset A of a space X.

(a) $A \in SPO\ (X)$.

(b) $A \subset \text{Cl}\ (\text{Int}\ (\text{Cl}\ (A)))$.

(c) $A \subset s\ \text{Int}\ (s\ \text{Cl}\ (A))$.

PROOF : This follows from Theorems 2.4 and 3.21 of Andrijević[1].

Definition 3.3— A function $f: X \to Y$ is said to be almost irresolute[10] if for each $x \in X$ and each semi-neighbourhood V of $f(x)$, $s\ \text{Cl}\ (f^{-1}\ (V))$ is a semi-neighbourhood of x.

Henceforth, "almost irresolute" always means "almost irresolute" in the sense of Definition 3.3, that is, Dube *et al.*[10].

Theorem 3.4— The following are equivalent for a function $f: X \to Y$:

(a) f is almost irresolute.

(b) $f^{-1}(V) \subset s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f^{-1}(V)))$ for every $V \in SO(Y)$.

(c) $f^{-1}(V) \subset \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(f^{-1}(V))))$ for every $V \in SO(Y)$.

(d) $f^{-1}(V) \in SPO(X)$ for every $V \in SO(Y)$.

PROOF : (a) $\Rightarrow$ (b) : Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. Since V is a semi-neighbourhood of $f(x)$, $s\ \mathrm{Cl}\,(f^{-1}(V))$ is a semi-neighbourhood of x and hence there exists $U \in SO(X, x)$ such that $U \subset s\ \mathrm{Cl}\,(f^{-1}(V))$. Therefore, we have $x \in U \subset s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f^{-1}(V)))$. This implies that $f^{-1}(V) \subset s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f^{-1}(V)))$.

(b) $\Rightarrow$ (a) : Let $x \in X$ and V be any semi-neighbourhood of $f(x)$. There exists $W \in SO(Y, f(x))$ contained in V. Therefore, we obtain

$$x \in f^{-1}(W) \subset s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f^{-1}(W))) \subset s\ \mathrm{Cl}\,(f^{-1}(W)) \subset s\ \mathrm{Cl}\,(f^{-1}(V)).$$

This implies that $s\ \mathrm{Cl}\,(f^{-1}(V))$ is a semi-neighbourhood of x.

It follows from Lemma 3.2 that (b), (c) and (d) are all equivalent.

Theorem 3.5— A function $f: X \to Y$ is almost irresolute if and only if $f(s\ \mathrm{Cl}\,(U)) \subset s\ \mathrm{Cl}\,(f(U))$ for every $U \in SO(X)$.

PROOF : *Necessity*— Let $U \in SO(X)$. Suppose that $y \notin s\ \mathrm{Cl}\,(f(U))$. There exists $V \in SO(Y, y)$ such that $V \cap f(U) = \phi$; hence $f^{-1}(V) \cap U = \phi$. Since $U \in SO(X)$, we have $s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f^{-1}(V))) \cap s\ \mathrm{Cl}\,(U) = \phi$. By Theorem 3.4, $f^{-1}(V) \cap s\ \mathrm{Cl}\,(U) = \phi$ and hence $V \cap f(s\ \mathrm{Cl}\,(U)) = \phi$. Therefore, we obtain $y \notin f(s\ \mathrm{Cl}\,(U))$. This shows that $f(s\ \mathrm{Cl}\,(U)) \subset s\ \mathrm{Cl}\,(f(U))$.

Sufficiency— Let $V \in SO(Y)$. Since $X - s\ \mathrm{Cl}\,(f^{-1}(V)) \in SO(X)$, we have $f(s\ \mathrm{Cl}\,(X - s\ \mathrm{Cl}\,(f^{-1}(V)))) \subset s\ \mathrm{Cl}\,(f(X - s\ \mathrm{Cl}\,(f^{-1}(V))))$ and hence

$$X - s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f^{-1}(V))) \subset f^{-1}(s\ \mathrm{Cl}\,(f(X - s\ \mathrm{Cl}\,(f^{-1}(V)))))$$

$$\subset f^{-1}(s\ \mathrm{Cl}\,(f(X - f^{-1}(V)))) \subset f^{-1}(s\ \mathrm{Cl}\,(Y - V)) = X - f^{-1}(V).$$

Therefore, we obtain $f^{-1}(V) \subset s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f^{-1}(V)))$. It follows from Theorem 3.4 that f is almost irresolute.

Theorem 3.6—The following are equivalent for a function $f: X \to Y$.

(a) f is almost irresolute.

(b) For each $x \in X$ and each $V \in SO(Y, f(x))$, there extsts $U \in SPO(X)$ containing x such that $f(U) \subset V$.

(c) $f^{-1}(F)$ is semi-preclosed in X for every semi-closed set F of Y.

(d) $\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(f^{-1}(B)))) \subset f^{-1}(s\ \mathrm{Cl}\,(B))$ for every subset B of Y.

(e) $f(\text{Int}(\text{Cl}(\text{Int}(A)))) \subset s\,\text{Cl}(f(A))$ for every subset A of X.

PROOF : (a) $\Rightarrow$ (b) : Let $x \in X$ and $V \in SO(Y, f(x))$. Set $U = f^{-1}(V)$, then by Theorem 3.4 U is a semi-preopen set containing x and $f(U) \subset V$.

(b) $\Rightarrow$ (a) : Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. There exists $U \in SPO(X)$ containing x such that $f(U) \subset V$. By Lemma 3.2, we obtain

$$x \in U \subset s\,\text{Int}(s\,\text{Cl}(U)) \subset s\,\text{Int}(s\,\text{Cl}(f^{-1}(V)))$$

and hence $f^{-1}(V) \subset s\,\text{Int}(s\,\text{Cl}(f^{-1}(V)))$. It follows from Theorem 3.4 that f is almost irresolute.

(a) $\Rightarrow$ (c) : This is obvious by Theorem 3.4.

(c) $\Rightarrow$ (d) ; Let B be any subset of Y. Since $s\,\text{Cl}(B)$ is semi-closed, $f^{-1}(s\,\text{Cl}(B))$ is semi-preclosed. By utilizing Lemma 3.2, we have

$$X - f^{-1}(s\,\text{Cl}(B)) \subset \text{Cl}(\text{Int}(\text{Cl}(X - f^{-1}(s\,\text{Cl}(B)))))$$

$$= X - \text{Int}(\text{Cl}(\text{Int}(f^{-1}(s\,\text{Cl}(B))))).$$

Therefore, we obtain $\text{Int}(\text{Cl}(\text{Int}(f^{-1}(B)))) \subset f^{-1}(s\,\text{Cl}(B))$.

(d) $\Rightarrow$ (e) : Let A be any subset of X. We have

$$\text{Int}(\text{Cl}(\text{Int}(A))) \subset \text{Int}(\text{Cl}(\text{Int}(f^{-1}(f(A))))) \subset f^{-1}(s\,\text{Cl}(f(A))).$$

Therefore, we obtain $f(\text{Int}(\text{Cl}(\text{Int}(A)))) \subset s\,\text{Cl}(f(A))$.

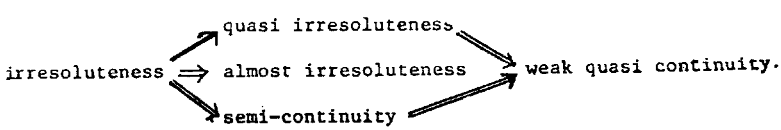
(e) $\Rightarrow$ (a) : Let $U \in SO(X)$. Since $s\,\text{Cl}(U) = U \cup \text{Int}(\text{Cl}(U)) = U \cup \text{Int}(\text{Cl}(\text{Int}(U)))$, we obtain $f(s\,\text{Cl}(U)) = f(U) \cup f(\text{Int}(\text{Cl}(\text{Int}(U)))) \subset f(U) \cup s\,\text{Cl}(f(U)) = s\,\text{Cl}(f(U))$. If follows from Theorem 3.5 that f is almost irresolute.

4. COMPARISONS

In this section, we investigate the relationship among irresoluteness, quasi irresoluteness, almost irresoluteness, semi-continuity and weak quasi continuity.

Definition 4.1— A function $f: X \to Y$ is said to be weakly quasi continuous[19] if for each $x \in X$, each open set U containing x and each open set V containing $f(x)$, there exists an open set G of X such that $\phi \neq G \subset U$ and $f(G) \subset \text{Cl}(V)$.

Remark 4.2 : The following implications hold for properties on a function :



It is obvious by Lemma 2.2 and Theorem 3.4 that irresoluteness implies almost irresoluteness. The other implications have been shown in section 7 of Di Maio and Noiri[8].

We shall show that quasi irresoluteness, almost irresoluteness and semi-continuity are respectively independent. The following example shows that almost irresoluteness does not imply weak quasi continuity and hence it implies neither quasi irresoluteness nor semi-continuity.

Example 4.3— Let X be the set of real numbers, τ the indiscrete topology for X and σ the discrete topology for X. Let $f: (X, \tau) \to (X, \sigma)$ be the identity function. Then f is almost irresolute but it is not weakly quasi continuous.

Example 4.4— Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f: (X, \tau) \to (X, \tau)$ be a function defined as follows : $f(a) = f(b) = a$ and $f(c) = c$. Then f is continuous and hence semi-continuous. However, f is not almost irresolute because there exists $\{b, c\} \in SO(X, \tau)$ such that $f^{-1}(\{b, c\}) \notin SPO(X, \tau)$.

By Examples 4.3 and 4.4, we observe that almost irresoluteness is independent of semi-continuity and also it is independent of weak quasi continuity. The following example and Example 4.3 show that almost irresoluteness and quasi irresoluteness are independent of each other.

Example 4.5—Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\phi, X, \{c\}\}$. Let $f: (X, \tau) \to (X, \sigma)$ be the identity function. Then f is quasi irresolute. However, it is not almost irresolute because there exists $\{c\} \in SO(X, \sigma)$ such that $f^{-1}(\{c\}) \notin PSO(X, \tau)$.

It has been shown in Examples 7.2 and 7.3 of Di Maio and Noiri[8] that quasi irresoluteness neither implies semi-continuity nor is implied by semi-continuity.

5. Semi-preopen Functions

In this section, we introduce the notion of semi-preopen functions which is independent of both notions of preopen functions and semi-open functions. It will be shown that every quasi irresolute function is almost irresolute if it has one of the following properties : "semi-preopen", "semi-open" and "preopen".

Definition 5.1—A function $f: X \to Y$ is said to be semi-preopen if $f(U) \in SPO(Y)$ for every $U \in SO(X)$.

Definition 5.2— A function $f: X \to Y$ is said to be semi-open[2] (resp. preopen[15]) if $f(U)$ is semi-open (resp. preopen) in Y for every open set U of X.

Rose[20] called preopen functions almost open and showed that $f: X \to Y$ is almost open if and only if $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$ for every open set V of Y. Therefore, "preopen" is equivalent to "almost open" in the sense of Wilansky[22].

We shall show that "semi-preopen", "preopen" and "semi-open" are respectively independent. It follows from Examples 1.8 and 1.9 of Noiri[17] that "preopen" and "semi-open" are independent of each other. The following example shows that "open" does not imply "semi-preopen" in general. Therefore, neither preopenness nor semi-openness implies semi-preopenness.

Example 5.3— Let $X = \{a, b, c, d\}$, $\tau = \{\phi, x, \{a\}, \{a, c\}\}$ and $\sigma = \{\phi, x, \{a\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is an open function. However, f is not semi-preopen because there exists $\{a, b\} \in SO(X, \tau)$ such that $f(\{a, b\}) \notin SPO(X, \sigma)$.

Let $f : (X, \tau) \to (X, \sigma)$ be the function of Example 4.3. Then f^{-1} is semi-preopen but not semi-open. Moreover, the following example shows that a semi-preopen function is not necessarily preopen.

Example 5.4— Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be a function defined as follows : $f(a) = a$, $f(b) = c$, and $f(c) = b$. Then f is semi-preopen. However, f is not preopen because there exists $\{a, b\} \in \tau$ such that $f(\{a, b\})$ is not preopen.

Theorem 5.5—The following are equivalent for a function $f : X \to Y$:

(a) f is semi-preopen.

(b) $f^{-1}(s\,\mathrm{Cl}(V)) \subset s\,\mathrm{Cl}(f^{-1}(V))$ for every $V \in SO(Y)$.

(c) $f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))) \subset s\,\mathrm{Cl}(f^{-1}(B))$ for every subset B of X.

(d) $f(\mathrm{Int}(A)) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f(A))))$ for every subset A of X.

(e) $f(U) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f(U))))$ for every $U \in SO(X)$.

(f) $f(U) \subset s\,\mathrm{Int}(s\,\mathrm{Cl}(f(U)))$ for every $U \in SO(X)$.

PROOF : (a) $\Rightarrow$ (b) : Let $V \in SO(Y)$ and $x \notin s\,\mathrm{Cl}(f^{-1}(V))$. There exists $U \in SO(X, x)$ such that $U \cap f^{-1}(V) = \phi$; hence $f(U) \cap V = \phi$. Therefore, we have $s\,\mathrm{Int}(s\,\mathrm{Cl}(f(U))) \cap V = \phi$ and hence $s\,\mathrm{Int}(s\,\mathrm{Cl}(f(U))) \cap s\,\mathrm{Cl}(V) = \phi$. Since f is semi-preopen, by Lemma 3.2 we obtain $f(U) \cap s\,\mathrm{Cl}(V) = \phi$ and $U \cap f^{-1}(s\,\mathrm{Cl}(V)) = \phi$. Therefore, we have $x \notin f^{-1}(s\,\mathrm{Cl}(V))$ and hence $f^{-1}(s\,\mathrm{Cl}(V)) \subset s\,\mathrm{Cl}(f^{-1}(V))$.

(b) $\Rightarrow$ (c) : Let B be any subset of Y. We obtain

$$f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))) = f^{-1}(s\,\mathrm{Cl}(\mathrm{Int}(B))) \subset s\,\mathrm{Cl}(f^{-1}(\mathrm{Int}(B)))$$
$$\subset s\,\mathrm{Cl}(f^{-1}(B)).$$

(c) $\Rightarrow$ (d) : Let A be any subset of X. Then, we have

$$X - f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f(A))))) = f^{-1}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(Y - f(A)))))$$
$$\subset s\,\mathrm{Cl}(f^{-1}(Y - f(A))) \subset s\,\mathrm{Cl}(X - A) = X - s\,\mathrm{Int}(A).$$

Therefore, we obtain $f(s\,\mathrm{Int}(A)) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f(A))))$.

(d) $\Rightarrow$ (e) : Let $U \in SO(X)$. We have $f(U) = f(s \text{ Int } (U)) \subset \text{Cl}(\text{Int}(\text{Cl}(f(U))))$.

(e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (a) : These follow from Lemma 3.2.

Theorem 5.6— *A* quasi irresolute function $f: X \to Y$ is almost irresolute if it satisfies one property of the following :

(a) semi-preopen, (b) semi-open, and (c) preopen.

PROOF : (a) Suppose that f is quasi irresolute and semi-preopen. Let $V \in SO(Y)$. It follows from Lemma 2.5 that $f^{-1}(V) \subset s \text{ Int } (f^{-1}(s \text{ Cl } (V)))$. Moreover, by Theorem 5.5 we obtain $f^{-1}(V) \subset s \text{ Int } (s \text{ Cl } (f^{-1}(V)))$. Therefore, it follows from Theorem 3.4 that f is almost irresolute.

(b) Suppose that f is quasi irresolute and semi-open. Let $V \in SO(Y)$. By Lemma 2.5, we have $f^{-1}(V) \subset s \text{ Int } (f^{-1}(s \text{ Cl } (V)))$. It is shown in Theorem 2 of Noiri[16] that f is semi-open if and only if $f^{-1}(s \text{ Cl } (B)) \subset \text{Cl}(f^{-1}(B))$ for every subset B of Y. Therefore, we obtain $f^{-1}(V) \subset s \text{ Int } (\text{Cl}(f^{-1}(V))) = \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(V))))$. By Theorem 3.4, f is almost irresolute.

(c) Suppose that f is quasi irresolute and preopen. Let $V \in SO(Y)$. By Lemma 2.5, we have $f^{-1}(V) \subset s \text{ Int } (f^{-1}(s \text{ Cl } (V))) \subset s \text{ Int } (f^{-1}(\text{Cl}(V)))$. Since f is preopen, by Theorem 11 of Rose[20] we have

$$f^{-1}(\text{Cl}(V) = f^{-1}(\text{Cl}(\text{Int}(V)) \subset \text{Cl}(f^{-1}(\text{Int}(V))) \subset \text{Cl}(f^{-1}(V)).$$

Therefore, we obtain $f^{-1}(V) \subset s \text{ Int } (\text{Cl}(f^{-1}(V))) = \text{Cl}(\text{Int}(\text{Cl}(f^{-1}(V))))$. By Theorem 3.4, f is almost irresolute.

Remark 5.7 : The function f in Example 4.3 is almost irresolute, open and semi-preopen. However, it is neither quasi irresolute nor semi-continuous.

6. Strongly Semi-closed Graphs

For a function $f: X \to Y$, the subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called a graph of f and is denoted by $G(f)$. Dube *et al.*[9] defined and investigated a strongly semi-closed graph. We shall improve on some results established by Dube *et al.*[9].

Definition 6.1—The graph $G(f)$ is said to be strongly semi-closed Dube[9] if for each, $(x, y) \notin G(f)$, there exists $U \in SO(X,x)$ and $V \in SO(Y,y)$ such that $[U \times s \text{ Cl } (V))] \cap G(f) = \phi$.

It follows immediately that $G(f)$ is strongly semi-closed if and only if for each $(x, y) \notin G(f)$, there exist $U \in SO(X, x)$ and $V \in SO(Y, y)$ such that $f(U) \cap s \text{ Cl } (V) = \phi$. The graph $G(f)$ is said to be semi-closed (resp. closed) if it is semi-closed (resp. closed) in the product space $X \times Y$. If $G(f)$ is strongly semi-closed, then it is semi-closed. However, the converse is false by Example 2 of Dube *et al.*[9].

A function $f: X \to Y$ is said to be pre-semi-open[6] if $f(U) \in SO(Y)$ for every $U \in SO(X)$. Every pre-semi-open function is both semi-open and semi-preopen. However, in Example 4.3, f^{-1} is semi-preopen but not semi-open and hence not pre-semi-open. Moreover, in Example 5.3, f is semi-open but not semi-preopen and hence not pre-semi-open.

Theorem 6.2— Let $f: X \to Y$ be either semi-preopen or semi-open. If $G(f)$ is closed, then $G(f)$ is strongly semi-closed.

PROOF : Since $G(f)$ is closed, for each $(x, y) \notin G(f)$, there exist open sets U and V containing x and y, respectively, such that $f(U) \cap V = \phi$. First, suppose that f is semi-preopen. Since V is open, we obtain $s\ \mathrm{Int}\,(s\ \mathrm{Cl}\,(f(U))) \cap s\ \mathrm{Cl}\,(V) = \phi$. Since f is semi-preopen, $f(U) \in SPO(Y)$ and hence $f(U) \cap s\ \mathrm{Cl}\,(V) = \phi$ by Lemma 3.2. This shows that $G(f)$ is strongly semi-closed. Next, suppose that f is semi-open. Since $f(U) \in SO(Y)$, we have $f(U) \cap s\ \mathrm{Cl}\,(V) = \phi$ and hence $G(f)$ is strongly semi-closed.

Corollary 6.3[9]— If $f: X \to Y$ is pre-semi-open and $G(f)$ is closed, then $G(f)$ is strongly semi-closed.

Noiri[18] showed that a semi-continuous function into a Hausdorff space has a semi-closed graph but it does not have a closed graph in general.

Theorem 6.4— If $f: X \to Y$ is weakly quasi continuous and Y is Hausdorff, then $G(f)$ is strongly semi-closed.

PROOF : For each $(x, y) \notin G(f)$, there exist disjoint open sets V and W of Y containing y and $f(x)$, respectively. Hence we have $\mathrm{Cl}\,(W) \cap \mathrm{Int}\,(\mathrm{Cl}\,(V)) = \phi$ and $\mathrm{Cl}\,(W) \cap s\ \mathrm{Cl}\,(V) = \phi$. Since f is weakly quasi continuous, by Theorem 4.1 of Noiri[18] there exists $U \in SO(X, x)$ such that $f(U) \subset \mathrm{Cl}\,(W)$. Therefore, we obtain $f(U) \cap s\ \mathrm{Cl}\,(V) = \phi$. This shows that $G(f)$ is strongly semi-closed.

Corollary 6.5[9]— A function $f: X \to Y$ has a strongly semi-closed graph if it has one property of the following :

(a) f is semi-continuous and Y is Hausdorff;

(b) f is weakly irresolute and Y is Urysohn;

(c) f is θ-irresolute and Y is Urysohn.

PROOF : This follows immediately from Theorems 2.6 and 6.4 and Remark 4.2.

A space X is said to be regular semi-T_2 Maheshwari *et al.*[13] (resp. semi-T_2 Maheshwari and Prasad[14] if for distinct points x, y of X, there exist disjoint regular semi-open (resp. semi-open) sets U and V such that $x \in U$ and $y \in V$.

Lemma 6.6—A space X is regular semi-T_2 if and only if X is semi-T_2.

PROOF : Every regular semi-T_2 space is obviously semi-T_2. Conversely, suppose that X is semi-T_2. For distinct points x, y of X, there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $U \cap V = \phi$; hence $U \cap s\,\mathrm{Cl}(V) = \phi$. By Lemma 2.2, we have $s\,\mathrm{Cl}(V) \in SO(X)$ and hence $s\,\mathrm{Cl}(U) \cap s\,\mathrm{Cl}(V) = \phi$. It follows from Lemmas 2.1 and 2.2 that $s\,\mathrm{Cl}(U)$ and $s\,\mathrm{Cl}(V)$ are regular semi-open. Therefore, X is regular semi-T_2.

Theorem 6.7— If $f: X \to Y$ is quasi irresolute and Y is semi-T_2, then $G(f)$ is strongly semi-closed.

PROOF : Let $(x, y) \notin G(f)$. Since Y is semi-T_2, there exist $y \in SO(Y, y)$ and $W \in SO(Y, f(x))$ such that $V \cap W = \phi$. By Lemma 2.2, $s\,\mathrm{Cl}(V) \cap s\,\mathrm{Cl}(W) = \phi$. Since f is quasi irresolute, there exists $U \in SO(X, x)$ such that $f(U) \subset s\,\mathrm{Cl}(W)$. Therefore, we obtain $f(U) \cap s\,\mathrm{Cl}(V) = \phi$. This shows that $G(f)$ is strongly semi-closed.

A surjection $f: X \to Y$ is said to be semi-s-connected[9] if $f^{-1}(V)$ is semi-clopen in X for every semi-clopen set V of Y. It follows from Lemma 2.5 that a surjection is set-s-connected if and only if it is quasi irresolute. Dube *et al.*[9] defined a space X to be extremally-s-disconnected if $s\,\mathrm{Cl}(U) \in SO(X)$ for every $U \in SO(X)$. However, by Lemma 2.2 the semi-closure of a semi-open set is always semi-open.

Corollary 6.8[9]— Let $f: X \to Y$ be a function and Y semi-T_2. Then the following hold :

(a) If f is irresolute, then $G(f)$ is strongly semi-closed.

(b) If f is a set-s-connected surjection and Y is extremally s-disconnected, then $G(f)$ is strongly semi-closed.

PROOF : This is an immediate consequence of Remark 4.2 and Theorem 6.7.

Corollary 6.9[21]— If $f: X \to Y$ is almost irresolute (in the sense of Thakur and Paik) and Y is regular semi-T_2, then $G(f)$ is semi-closed.

PROOF : By Theorem 2.6, "quasi irresolute" is equivalent to "almost irresolute" in the sense of Thakur and Paik[21]. Therefore, this result is an immediate consequence of Lemma 6.6 and Theorem 6.7.

Finally, we shall obtain a sufficient condition for a function to be quasi-irresolute. A subset S of a space X is said to be s-closed relative to X, Di Maio and Noiri[7] if for every cover $\{V_\alpha \mid \alpha \in \nabla\}$ of S by semi-open sets of X, there exists a finite subset ∇_0 of ∇ such that $S \subset \cup \{s\,\mathrm{Cl}(V_\alpha) \mid \alpha \in \nabla_0\}$. If $S = X$, then the space X is said to be s-closed[7]. A space X is said to be extremally disconnected if the closure of every open set of X is open in X.

Lemma 6.10— Let X be extremally disconnected and $f: X \to Y$ have a strongly semi-closed graph. If K is s-closed relative to Y, then $f^{-1}(K)$ is semi-closed in X.

Proof : Let $x \notin f^{-1}(K)$. For each $y \in K$, $(x, y) \notin G(f)$ and hence there exist $U(y) \in SO(X, x)$ and $V(y) \in SO(Y, y)$ such that $f(U(y)) \cap s\,\mathrm{Cl}(V(y)) = \phi$. Therefore, we have $U(y) \cap f^{-1}(s\,\mathrm{Cl}(V(y))) = \phi$ for each $y \in K$. Since $\{V(y) \mid y \in K\}$ is a cover of K by semi-open sets of Y, there exists a finite number of points $y_1, y_2, \ldots, y_n$ in K such that $K \subset \cup \{s\,\mathrm{Cl}(V(y_i)) \mid i = 1, 2, \ldots, n\}$. Put $U = \cap \{U(y_i) \mid i = 1, 2, \ldots, n\}$. Since X is extremally disconnected, it follows from Theorem 2.9 of Janković[11] that $U \in SO(X, x)$. Moreover, we have $U \cap f^{-1}(K) = \phi$ and hence $x \notin s\mathrm{Cl}(f^{-1}(K))$. This implies that $f^{-1}(K)$ is semi-closed in X.

Theorem 6.11— Let X be extremally disconnected and Y s-closed. If $f : X \to Y$ has a strongly semi-closed graph, then it is quasi irresolute.

Proof : Let $V \in SO(Y)$. By Lemma 2.2, $s\,\mathrm{Cl}(V)$ is semi-clopen. It follows from Propositions 2.3 and 4.2 of Di Maio and Noiri[7] that $s\,\mathrm{Cl}(V)$ is s-closed relative to Y. By Lemma 6.10, $f^{-1}(s\,\mathrm{Cl}(V))$ is semi-closed in X. Therefore, we obtain $s\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(s\,\mathrm{Cl}(V))$. It follows from Lemma 2.5 that f is quasi irresolute.

Corollary 6.12—Let X be extremally disconnected and Y s-closed semi-T_2. A function $f : X \to Y$ is quasi irresolute if and only if $G(f)$ is strongly semi-closed.

Proof : This is an immediate consequence of Theorem 6.7 and 6.11.

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