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ON SOME DUAL INTEGRAL EQUATIONS INVOLVING BESSEL FUNCTION OF ORDER ONE

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Three different sets of dual integral equations, involving Bessel function of order one, arising in some special axi-symmetric problems of elasticity theory and viscous flow theory are handled for their solutions, either in closed form or in terms of Fredholm integral equations of the second kind. The final results are in full agreement with the ones obtained through the Green's function technique, used previously, for solving the mixed boundary value problems considered here.

1. INTRODUCTION

A number of workers have studied mixed boundary value problems, in the axisymmetric case, arising in Elasticity theory associated with punches and cracks[1–3] and in the theory of viscous flows induced by steady rotation or harmonic oscillation of a circular disc[4–8]. Of all possible methods of solution of these axi-symmetric problems, the method of reduction to a set of dual integral equations[1], appears to be the most natural and straightforward method of attack. But, as pointed out by Shail[3,5] and demonstrated by Stallybrass[9], certain dual integral equations are not amenable to their solution straightaway. It is for this difficulty that Stallybrass[8] and Shail[3,5,6] have utilised an integral representation of the principal unknown potential by employing a Green's function technique devised specially for the purpose.

We have shown in the present paper that by a suitable use of the Bessel equation itself, it is possible to handle all the problems treated previously via the dual integral equations only and that the details of the Green's function technique can be avoided, if we assume throughout the analysis that $\int_0^\infty p(s)\, J_1(sr)\, ds = \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon s}\, p(s)\, J_1(sr)\, ds$, wherever such integrals occur. It must be emphasised that the Green's function technique is better-suited to problems associated with more general axi-symmetric bodies than the circular disc for which the dual integral equations are the superior ones.

2. THE DUAL INTEGRAL EQUATIONS AND THEIR SOLUTIONS

Problem 1

The dual integral equations

$$\int_0^\infty s^2 A(s)\, J_1(sr)\, ds = -\sigma(r),\ 0 < r < a \qquad \ldots(1)$$

$$\int_0^\infty sA(s)\, J_1(sr)\, ds = 0, \quad r > a \qquad \ldots(2)$$

arise (see Erguven[9]) in the study of a static penny-shaped crack problem in a homogeneous isotropic elastic solid under torsion. Here $\sigma(r)$ represents the distribution of the shear-stress on the face of the crack and it is required that the displacement field given by the integral on the left of eqn. (1b) is zero at $r = a$, for the purpose of continuity.

The method of solution of the equations (1) and (2) is as described below, and is different from the method described in Sneddon's book[1].

We set

$$\int_0^\infty sA(s)\, J_1(sr)\, ds = f(r),\ 0 < r < a. \qquad \ldots(3)$$

Then, using the well-known Hankel's inversion formula to the eqns. (2) and (3), we obtain

$$A(s) = \int_0^a \lambda f(\lambda)\, J_1(\lambda s)\, d\lambda. \qquad \ldots(4)$$

The equations (1) and (4) finally give rise to the following integral equation for the unknown function $f(\lambda)$:

$$\int_0^\infty s^2 J_1(sr)\, ds \int_0^a \lambda f(\lambda)\, J_1(s\lambda)\, d\lambda = -\sigma(r),\ (0 < r < a) \qquad \ldots(5)$$

after interchanging the orders of integration, assuming that such an interchange is permissible here and even later on, for the other two problems treated in this paper.

If we next use the idea that the Bessel function $J_1(sr)$ satisfies the ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} + s^2 \right) J_1(sr) = 0 \qquad \ldots(6)$$

we observe that under certain special circumstances (as applicable to the crack problem mentioned above), we can rewrite the integral equation (5) in the form

$$\left(r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} - 1 \right) u(r) = r^2 \sigma(r),\ (0 < r < a) \qquad \ldots(7)$$

with

$$u(r) = \int_0^\infty J_1(sr)\, ds \int_0^a \lambda f(\lambda)\, J_1(\lambda s)\, d\lambda. \qquad \ldots(8)$$

we then solve the ODE (7) by the standard method of variation of parameters and obtain the general solution in the form

$$u = c_1^0\, r + \frac{c_2^0}{r} + \frac{r}{2}\int^r \sigma(t)\, dt - \frac{1}{2r}\int^r t^2 \sigma(t)\, dt \qquad \ldots(9)$$

where c_1^0 and c_2^0 are arbitrary constants to be determined from the physical considerations of the problem.

Of these two arbitrary constants, the constant c_2^0 must be chosen to be zero in order that u is finite at $r = 0$, and the constant c_1^0 will be settled a little later on.

Next, interchanging the orders of integration in eqn. (9), and using the result (see Shail[5]),

$$\int_0^\infty J_1(sr)\, J_1(s\lambda)\, ds = \frac{2}{\pi\lambda r}\int_0^{\min(r,\lambda)} \frac{v^2\, dv}{[(r^2 - v^2)(\lambda^2 - v^2)]^{1/2}}\,. \qquad \ldots(10)$$

We obtain

$$\frac{2}{\pi}\int_0^r \frac{v^2\, dv}{(r^2 - v^2)^{1/2}} \int_v^a \frac{f(\lambda)\, d\lambda}{(\lambda^2 - v^2)^{1/2}} = r u(r),\ (0 < r < a). \qquad \ldots(11)$$

Using the Abel's inversion formulae[1], repeatedly to equation (11), we find that

$$v\int_v^a \frac{f(\lambda)\, d\lambda}{(\lambda^2 - v^2)^{1/2}} = \int_0^v \frac{d}{dt}(tu(t))/(v^2 - t^2)^{1/2}\, dt,$$

and, hence,

$$f(\lambda) = \frac{2\lambda}{\pi a (a^2 - \lambda^2)^{1/2}}\int_0^a \frac{\frac{d}{dt}(tu(t))\, dt}{(a^2 - t^2)^{1/2}} - \frac{2\lambda}{\pi}\int_\lambda^a \frac{\left(\frac{dn}{dt}\right) dt}{(t^2 - \lambda^2)^{1/2}} \qquad \ldots(12)$$

where

$$n(t) = \frac{1}{t}\int_0^a \frac{\frac{d}{dm}(mu(m))}{(t^2 - m^2)^{1/2}}\, dm. \qquad \ldots(13)$$

Equations (12) and (13), along with eqn. (9) completely solve the integral equation (5), if the constant c_1^0 is determined. In order to determine the constant c_1^0, we need to use the result (12) and the observation that $f(a) = 0$, arising out of the physical requirement involving the continuity of $f(r)$ at $r = a$, as argued earlier. We then obtain the equation that

$$\int_0^a \frac{\frac{d}{dt}(tu(t))\,dt}{(a^2 - t^2)^{1/2}} = 0 \qquad \ldots(14)$$

and this, together with eqn. (9) serves as the determining equation for the constant c_1^0.

As a particular case, if we take $\sigma(r) = cr$, where c is a known constant, as considered by Erguven[10], we easily find that

$$u(r) = c_1^0 + \frac{c}{8} r^2 \qquad \ldots(15)$$

and eqn. (14) gives that

$$c_1^0 = -\frac{c}{6} a^2 \qquad \ldots(16)$$

so that eqn. (12) decides that

$$f(\lambda) = \frac{-4c}{3\pi} \lambda (a^2 - \lambda^2)^{1/2} \qquad \ldots(17)$$

which agrees with the result of Erguven[10].

Problem 2

The following dual integral equations arise in the study of the dynamic Reissner-Sagoci problem considered by Shail[3] :

$$\int_0^\infty c(\xi) J_1(\xi\rho)\,d\xi = \frac{\alpha\rho}{p}, \; (0 \leqslant \rho < 1) \qquad \ldots(18)$$

$$\int_0^\infty (\xi^2 + p^2/\beta^2)^{1/2} C(\xi) J_1(\xi\rho)\,d\xi = 0, \; (\rho > 1). \qquad \ldots(19)$$

In order to derive the above dual equations, we have used the representation of the solution of the mixed boundary value problem of Shail[3], in the form :

$$\bar{v}(\rho, z) = \int_0^\infty C(\xi)\, e^{-(p^2/\beta^2+\xi^2)^{1/2} z} J_1(\xi\rho)\,d\xi, \quad \left\{\begin{matrix}\rho \geqslant 0\\ z > 0\end{matrix}\right\} \qquad \ldots(20)$$

which must satisfy the p.d.e. and the boundary conditions as given by

$$\frac{\partial^2 \bar{v}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\bar{v}}{\partial\rho} - \frac{\bar{v}}{\rho^2} + \frac{\partial^2\bar{v}}{\partial z^2} - \frac{p^2\bar{v}}{\beta^2} = 0, \; z > 0 \qquad \ldots(21)$$

and

$$\left.\begin{aligned} \bar{v} &= \frac{\alpha\rho}{p}, (0 \leqslant \rho < 1) \\ \frac{\partial \bar{v}}{\partial t} &= 0, (\rho > 1) \end{aligned}\right] \text{on } z = 0. \qquad \ldots(22)$$

Shail[3] has used, like Stallybrass[9], a special Green's function technique to solve the mixed problems (21) and (22), as no direct method of attack exists to solve the dual eqns. (18) and (19). We present an approach, which is similar to the one employed for Problem 1 above, utilizing the Bessel equation (6), and show that for large values of p, the solution of the dual equations (18) and (19) can be determined exactly in the same manner as described by Shail[3].

We set

$$C(\xi) = \left(\frac{p^2}{\beta^2} + \xi^2\right)^{1/2} D(\xi) \qquad \ldots(23)$$

and rewrite eqns. (18) and (19) in the form

$$\int_0^\infty \left(\frac{p^2}{\beta^2} + \xi\right)^{1/2} D(\xi)\, J_1(\xi\rho) d\xi = \frac{\alpha\rho}{p}, (0 < \rho < 1) \qquad \ldots(23)$$

and

$$\int_0^\infty \left(\frac{p^2}{\beta^2} + \xi^2\right) D(\xi)\, J_1(\xi\rho)\, d\xi = 0\ (\rho > 1). \qquad \ldots(25)$$

Equation (25) can be recast in the form

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2} - \frac{p^2}{\beta^2}\right)\int_0^\infty D(\xi)\, J_1(\xi\rho)\, d\xi = 0, (\rho > 1) \qquad \ldots(26)$$

after using the Bessel equation (6), and this, on integration gives that

$$\int_0^\infty D(\xi)\, J_1(\xi\rho)\, d\xi = C_0\, K_1(p\rho/\rho), (\rho > 1) \qquad \ldots(27)$$

after neglecting the part of the solution involving $I_1\ (p\rho/\beta)$, where $I_1(x)$ and $K_1(x)$ are the modified Bessel functions of the first order and C_0 is an arbitrary constant to be determined by using order physico-mathematical consideration.

We thus find that the dual equations (24) and (25) can be recast into the form (24) and (27) respectively.

If we then set

$$\int_0^\infty (p^2/\beta^2 + \xi^2)^{1/2} D(\xi) J_1(\xi\rho)\, d\xi = g(\rho), (\rho > 1) \qquad ...(28)$$

and use the Hankel inversion formula to eqns. (24) and (28), we obtain that

$$(p^2/\beta^2 + \xi^2)^{1/2} D(\xi) = \frac{\alpha}{p} J_2(\xi) + \xi \int_1^\infty \lambda\, g(\lambda) J_1(\xi\lambda)\, d\lambda \qquad ...(29)$$

(See Gradshteyn and Ryzhik[11], p. 683) and that eqn. (27) finally gives rise to the following integral equation for the unknown function $g(\rho)$:

$$\int_1^\infty \lambda\, g(\lambda) M(\rho, \lambda)\, d\lambda = C_0 K_1\left(\frac{p\rho}{\beta}\right) - \frac{\alpha}{p}\int_0^\infty \frac{J_2(\xi) J_1(\xi\rho)}{(\xi^2 + p^2/\beta^2)^{1/2}}\, d\xi,$$

$$(\rho > 1) \qquad ...(30)$$

where

$$M(\rho, \lambda) = \int_0^\infty \frac{\xi}{(\xi^2 + p^2/\beta 1)^{1/2}} J_1(\xi\rho) J_1(\xi\lambda)\, d\xi. \qquad ...(31)$$

The integral eqation (30) is similar to the one obtained by Shail[3], and can be attacked for solution, for large p, by a repeated use of the Abel's inversion formulae as described by Shail[3], if the following asymptotic result for the kernel $M(\rho, \lambda)$ is made use of :

$$M(\rho, \lambda) \sim \frac{1}{\pi (\rho\lambda)^{1/2}} e^{q(\rho+\lambda)} \int_{\max(\rho\lambda)}^\infty \frac{e^{-2qw}\, dw}{(w-\rho)^{1/2} (w-\lambda)^{1/2}} + O(q^{-2}),$$

$$...(32)$$

with $q = p/\beta$.

We do not proceed any further with this problem here, as the other details are going to be repetitions of Shail's work[3].

Problem **3**

The following dual integral equations arise, as shown by Goodrich[4], in the study of a viscous flow problem induced by a rotating circular disc, kept on the surface of a bulk fluid of viscosity μ, which is otherwise contaminated by an adsorbed fluid film of different viscosity η.

$$\int_0^\infty f(y) J_1(yr)\, dy = \omega r, (0 \leqslant r < a) \qquad ...(33)$$

$$\int_0^\infty (\mu y + \eta y^2) f(y)\, J_1(yr)\, dy = 0, \; (r > a) \qquad ...(34)$$

where ω is the constant angular velocity of the disc of radius a, rotating around its axis.

Goodrich[4] has devised a special method of solving the above dual equations (33) and (34) in the three different circumstances, as given by the cases (i) $\mu = 0$, (ii) $\eta = 0$ and (iii) $\mu \neq 0$, $\eta \neq 0$. But, as pointed out by Shail[5], Goodrich's solutions are not the correct ones, since they involve certain divergent integrals.

It is because of this major difficulty that Shail[5] has attacked the physical problem of Goodrich[4] and several other generalization of it[6-8], by a method utilizing the Green's function technique.

We have shown below that the dual integral equations (33) and (34) can also be attacked for their solution in the three cases (i), (ii) and (iii) as considered by Goodrich[4], in a straightforward manner as described for the Problems 1 and 2 above, and we thus infer that the use of the Green's function technique can be avoided here also.

Case (i) : $\mu = 0$— In this special case, the dual equations (33) and (34) take up the forms :

$$\int_0^\infty f(y)\, J_1(yr)\, dy = \omega r \; (0 \leqslant r \leqslant a) \qquad ...(35)$$

$$\int_0^\infty y^2 f(y)\, J_1(yr)\, dy = 0 \; (r > a) \qquad ...(36)$$

and, the second eqn. (36), can be recast, by using the Bessel equation (1*f*), in the form

$$\int_0^\infty f(y)\, J_1(yr)\, dy = C_0/r \; (r > a) \qquad ...(37)$$

where C_0 is an arbitrary constant to be determined.

A straightforward use of the Hankel inversion formula to eqn. (35) and (37) gives

$$f(y) = C_0\, J_0(ay) + \omega a^2\, J_2(ay). \qquad ...(38)$$

(see Gradshteyn and Ryzhik[11], p. 683).

We ultimately find that the constant C_0 appearing in (38) must be chosen to be ωa^2 in order to make the integral in (36) convergent, in the sense mentioned in the introduction, and this agrees with the observation of Goodrich, even though $f(y)$ is different.

Case (*ii*) : $\eta = 0$— In this case, the dual equations to be be solved are the ones as given by eqn. (35) and the new equation

$$\int_0^\infty y f(y) J_1(yr)\, dy = 0, \ (r > a). \qquad \ldots(39)$$

Assuming that

$$\int_0^\infty y f(y) J_1(yr)\, dy = g(r), \ (0 \leqslant r \leqslant a) \qquad \ldots(40)$$

and using Hankel's inversion formula we find that

$$f(y) = \int_0^\infty g(\lambda)\, \lambda\, J_1(\lambda y)\, dy. \qquad \ldots(41)$$

Then using (41) in eqn. (35) we ultimately derive that

$$\frac{2}{\pi r} \int_0^r \frac{v^2\, dv}{(r^2 - v^2)^{1/2}} \int_v^a \frac{g(\lambda)\, d\lambda}{(\lambda^2 - v^2)^{1/2}} = \omega r, \ (0 \leqslant r \leqslant a) \qquad \ldots(42)$$

obtained after utilizing the formula (10).

A repeated Abel inversion procedure, like the one adopted in the previous problems, ultimately gives

$$\pi\, g(r) = 4\omega r\, (a^2 - r^2)^{-1/2}, \ (0 \leqslant r \leqslant a) \qquad \ldots(43)$$

and the solution of the dual equations can be completed by using (41).

Case (*iii*) : $\mu \neq 0$, $\eta \neq 0$— In the most general case of the dual equations (33) and (34), when neither μ nor η is zero, we obtain a Fredholm integral equation of the second kind, which is similar to the one obtained by Shail[5], by means of a procedure as described below.

We first rewrite eqns. (33) and (34) in the form

$$\int_0^\infty f(y) J_1(yr)\, dy = \omega r \ (0 \leqslant r \leqslant a) \qquad \ldots(44)$$

and

$$\int_0^\infty y\, (1 + \lambda_0\, ay) f(y) J_1(yr)\, dy = 0 \ (r > a) \qquad \ldots(45)$$

where $\lambda_0 = \eta/\mu a$.

Then, setting as in the previous problems,

$$\int_0^{\infty} y\,(1 + \lambda_0\, ay)\, f(y)\, J_1(yr)\, dy = g(r) \qquad (0 \leqslant r \leqslant a) \qquad \ldots(46)$$

and using Hankel's inversion formula to eqns. (46) and (45), we find that

$$f(y) = \frac{1}{(1 + \lambda_0\, ay)} \int_0^{a} \lambda\, g(\lambda)\, J_1(y\lambda)\, d\lambda. \qquad \ldots(47)$$

We next use the relation (47) in (44) and interchange the orders of integration to obtain the following integral equation for the function $g(r)$:

$$\int_0^{a} \lambda\, g(\lambda)\, d\lambda \int_0^{\infty} \left(\frac{1}{1 + \lambda_0\, ay}\right) J_1(yr)\, J_1(y))\, dy = \omega r,\ (0 \leqslant r \leqslant a). \quad \ldots(48)$$

Equation (48) is an integral equation of the first kind and it can be converted into an integral equation of the second kind, by observing first that

$$\frac{1}{1 + \lambda_0\, ay} = \tfrac{1}{2}\left[1 + \frac{1 - \lambda_0\, ay}{1 + \lambda_0\, ay}\right] \qquad \ldots(49)$$

and then using the result (10) along with an Abel's inversion procedure, of a type similar to what is known as Williams's method[12].

We find that the second kind Fredholm equation is obtained finally in the form

$$g^*(v) + \int_0^{a} L^*(t, v)\, g^*(t)\, dt = 4\omega v,\ (0 \leqslant v \leqslant a) \qquad \ldots(50)$$

where

$$g^*(v) = v \int_v^{a} \frac{q(\lambda)\, d\lambda}{(\lambda^2 - v^2)^{1/2}}, \qquad \ldots(51)$$

and

$$L^*(t,v) = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \lambda_0\, ay}{1 + \lambda_0\, ay} \sin(ty) \sin(vy)\, dy. \qquad \ldots(52)$$

Equation (50) can be easily identified to be similar to the one obtained by Shail[5], for the problem of Goodrich[4].

A slightly more general dual integral equations, than the ones given by (33) and (34), which are

$$\int_0^\infty \left[1 + \frac{1 - \lambda_0\, ay}{1 + \lambda_0\, ay} e^{-2hy} \right] f(y)\, J_1(y\rho)\, dy = \omega\rho \; (0 \leqslant \rho \leqslant a) \qquad \ldots(53)$$

and

$$2 \int_0^\infty f(y)\, y\, J_1(y\rho)\, dy = 0, (\rho > a) \qquad \ldots(54)$$

where $h > 0$, can also be reduced to a Fredholm integral equation of the second kind by a method similar to the one described above.

The equations (53) and (54) arise in the rotating disc problem of Shail[6], when the disc is kept at a distance a below the contaminated surface considered by Goodrich[4], so that the particular case $h = 0$ of (53) and (54) correspond to eqns. (33) and (34).

3. Conclusion

The present paper has dealt with the dual-integral-equations-formulation of some of the well-studied axi-symmetric mixed boundary value problems of potential theory in a manner different from the ones used in the literature, but is very straightforward otherwise. The principal aim has been to show that for the three problems discussed here, or its generalizations, it is not necessary to employ any other complicated technique, such as the Green's function technique, used by previous workers even though the merit of the latter technique is unquestionably of a higher level, if one has to handle axisymmetric bodies other than Circular discs as has been the case with the above three problems.

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