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BANACH SPACE VALUED DISTRIBUTIONAL MELLIN TRANSFORM AND FORM INVARIANT LINEAR FILTERING

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From invariant filters are those shift variant filters such that a linear scaling of their input gives rise to a linear scaling of their output. In this work we develop a theory of Mellin transform and Mellin type convolution for Banach space valued distributions. Application of this theory to relate the input, output singals of the form invariant systems is given.

1. Introduction

Any physical system, be it electrical or optical, can be characterized by an input signal, output signal and system function. Time invariance (shift invariance) is a property possessed by many physical systems. Zemanian[1,2] introduced the concept of Banach space valued distributions. He presented a theory for the convolution and Laplace transformation of Banach space valued distributions and used these concepts for convolution representation of continuous linear time invariant systems.

The form invariant filters are those filters, such that a linear scaling of their input gives rise to a linear scaling (generally through a different scaling factor) of their output. Braccini and Gambardella[3,4] defined the form invariance property and obtained the most general class of linear shift variant systems satisfying it. They related the input and output signals of the form invariant systems by using Mellin transform. Further they discussed applications of form invariant filters in optical pattern recognition, image restoration, image reconstruction, radar signal processing etc. Braccini and Gambardella[4] considered the most general input output relationship for a linear one dimensional system as

$$g(x) = \int_{-\infty}^{\infty} f(t)\, w(t, x)\, dt$$

where $f(t)$ is the input signal, $g(x)$ is the output signal, and $w(t, x)$ is the kernel of the integral transform. The integral representation has no sense when the input signal $f(t)$ is a singularity signal. For example singularity signal like δ cannot be treated as Lebesgue integrable function (LIF). If, we treat δ as LIF we get senseless results (see Gelfand and Shilov[5], p. 4, Zemanian[6], p. 10). The theory of distributions (generalized functions) provides a natural language for such signals (see Zemanian[7], Wohlers and

Beltrami[8]). By using distribution theory we can generalize and correct results of of Braccini and Gambardella[4].

Recently the author[9-11] extended some integral transforms to Banach space valued distributions and discussed their applications to system theory.

In this paper we develop a theory of Mellin transform and Mellin type convolution for Banach space valued distributions. We use this theory to relate the input, output signals of the form invariant systems. Justification for these studies includes the desire to admit a larger class of systems and input output pairs. When the input signal $f(t)$ is a LIF our results of section 5 reduce to results of Braccini and Gambardella[4].

The plan of the paper is as follows. In section 3, we develop a theory for Mellin transform of Banach space valued distributions. In section 4 we present a theory for Mellin type convolution of Banach space valued distributions. Finally in section 5, we discuss Banach space valued distributional form invariant linear system and use Mellin transform to relate the input output signals of the form invariant systems.

2. Notations and Terminology

The notations and terminology of this work will follow that of Tiwari[9,10,11] and Zemanian[1,2]. R and $\mathbb{C}$ denote, respectively, the real line and complex plane R_+ denotes the positive half-line $\{t \in R : 0 < t < \infty\}$. A and B denote complex Banach spaces. $E(A)$ denotes the linear space of all smooth functions ϕ from R into A. $E_+(A)$ is linear space of all smooth functions from R_+ into A. Let K be a compact subset of R. $D_K(A)$ denotes the linear space of all smooth, A valued functions ϕ such that supp $(\phi) \subset K$. The space $D_K(A)$ is assigned the topology generated by the collection $\{r_k\}_{k=0}^{\infty}$ of seminorms, where

$$r_k(\phi) = \sup_{t \in K} \|\phi^k(t)\|_A.$$

$\|\cdot\|_A$ denotes the norm in A. Let $\{K_j\}_{j=1}^{\infty}$ be a sequence compact subsets in R such that $K_1 \subset K_2 \subset \ldots, \bigcup_{j=1}^{\infty} K_j = R$. The space $D(A)$ is defined as the inductive limit space generated by $D_{K_j}(A)$. That is

$$D(A) = \operatorname*{ind}_{j \to \infty} D_{K_j}(A).$$

Similarly we have

$$D_+(A) = \operatorname*{ind}_{j \to \infty} D_{K_j}(A)$$

here $K_1, K_2 \ldots,$ are compact subsets of R_+ and $R_+ = \bigcup_{j=1}^{\infty} K_j$

For $A = \mathbb{C}$, we write $E(A) = E$, $E_+(A) = E_+$, $D(A) = D$ and $D_+(A) = D_+$. When U and V are topological spaces, $[U; V]$ denotes the space of all continuous linear mappings of U into V. The symbol $<f, \phi>$ denotes the element of V assigned to $\phi \in U$ by $f \in [U; V]$. The notation □ denotes the end of a proof.

3. Mellin Transformation

Following Tiwari[12] we define the space $M_{a,b,\alpha}(A)$.

3.1. *The Space*

$M_{a,b,\alpha}(A)$

For $\alpha \geqslant 0$, we define

$$M_{a,b,\alpha}(A) = \{\phi : \phi \in E_+(A),$$

$$i_{a,b,k}(\phi) = \sup_{\mathbb{R}_+} \|\lambda_{a,b}(t)\, t^{k+1}\, D_t^k\, \phi(t)\|$$

$$\leqslant C_k L^k k^{k\alpha},\ k = 0, 1, \ldots\}.$$

The constants L and C_k depend on the function ϕ and

$$\lambda_{a,b}(t) = \begin{cases} t^{-a} & 0 < t \leqslant 1 \\ t^{-b'} & 1 < t < \infty. \end{cases}$$

For $k = 0$, we set $k^{k\alpha} = 1$. The topology of the space $M_{a,b,\alpha}(A)$ is generated by the family of seminorms $\{i_{a,b,k}\}_{k=0}^{\infty}$. The space $M_{a,b}(A)$ is defined as the inductive limit space generated by $M_{a,b,\alpha}(A)$. That is

$$M_{a,b}(A) = \operatorname*{ind}_{\alpha \to \infty} M_{a,b,\alpha}(A).$$

Following Zemanian[1] (p. 102) we define the space $M(w, z; A)$ as

$$M(w, z; A) = \operatorname*{ind}_{n \to \infty} M_{a_n b_n}(A)$$

where

$$a_n \to w_+,\ b_n \to z_-,\ w, z \in [-\infty, \infty].$$

Let B be a complex Banach space. Any $f \in [M(w, z; A), B]$ is a Banach space-valued distribution. When $A = B = \mathbb{C}$, f becomes a scalar valued distribution.

Let S be a collection of bounded subsets of $M_{a,b}(A)$. The topology of uniform convergence on $[M_{a,b}(A), B]$ is that generated by the collection of seminorms $\{\sigma_\Omega\}_{\Omega \in S}$ where,

$$\sigma_\Omega(f) = \sup_{\phi \in \Omega} \| <f, \phi> \|_B.$$

$f \in [M_{a,b}(A), B], \Omega \in S.$

The weak topology or simple topology for $[M_{a,b}(A), B]$ is generated by the collection seminorms $\{\rho_\phi\}$, where ϕ traverses $M_{a,b}(A)$ and

$$\rho_\phi(f) = \| < f, \phi > \|_B.$$

We now define Mellin transform of A-valued and $[A, B]$ valued distributions (see Zemanian[1], p. 115).

3.2. Mellin Transform of A-valued Distributions

Let $f \in [D_+; A]$. We way that f is Mellin transformable if there exists two members $\sigma_1, \sigma_2 \in [-\infty, \infty]$ such that $\sigma_1 < \sigma_2, f \in [M(\sigma_1, \sigma_2): A]$, and in addition $f \notin [M(w, z); A]$ if either $w < \sigma_1$ or $z > \sigma_2$. With $\Omega_f = \{s : \sigma_1 < \text{Re}(s) < \sigma_2\}$, $t^{s-1} \in M(\sigma_1, \sigma_2)$ we define Mellin transform Mf of f as

$$F(s) = (Mf)(s) = < f(t), t^{s-1}>, \; s \in \Omega_f. \quad \text{..(3.1)}$$

It can be easily proved that $F(s)$ is an A-valued analytic function on Ω_f.

The space $[M(w, z; A); B]$ can be identified with the space $[M(w, z); (A; B)]$ through the equation

$$< f_y, \phi > a = < y, \phi a >$$

(see Zemanian[1], p. 105) where $f_y \in [M(w, z); [A; B]]$,

$y \in [M(w, z; A); B], \phi \in M(w, z)$ and $a \in A$.

Because of the above identification we use the same symbol to denote both f_y and y, and define Mellin transform of $[A; B]$ valued distribution as :

We say that $y \in [D_+; A]$ is Mellin transformable if there exists $\eta_1, \eta_2 \in [-\infty, \infty]$ such that $\eta_1 < \eta_2, y \in [M(\eta_1, \eta_2; A); B]$ and $y \notin [M(w,z; A); B]$ if either $w < \eta_1$ or $z > \eta_2$.

Using the above identification $y \in [M(\eta_1, \eta_2); [A; B]]$ also. Hence we now define the Mellin transform Y of y as

$$Y(s) = < y(t); t^{s-1} >, s \in \Omega_y \quad \text{...(3.2)}$$

where $\Omega_y = < y(t); t^{s-1} >, s \in \Omega_y$ ...(3.2)

where $\Omega_y = \{s : \eta_1 < \text{Re}(s) < \eta_2\}$ is called strip of definition for the Mellin transform of y.

Theorem 3.1 (*Analyticity Theorem*)—If $My = Y(s)$ for $s \in \Omega_y$, then $Y(s)$ is an $[A; B]$ valued analytic function and for each non negative integer k

$$Y^k(s) = < y(t), D_s^k \; t^{s-1} >$$

PROOF : From definition the result is true for $k = 0$. With fixed s and $\Delta s \neq 0$. consider

$$< y(t), \psi_{\Delta s}(t) >$$
$$= \frac{Y^k(s+\Delta s) - Y^k(s)}{\Delta s} - < y(t), D_s^k t^{s-1} >.$$

It is not difficult to show that $\psi_{\Delta s}(t)$ converges to zero in $M(\eta_1, \eta_2)$ as $\Delta s \to 0$.

4. Mellin Type Convolution

Theorem 1.1—The scaling operator $S_a : \phi(t) \to \phi(at)$, $a > 0$ is a topological automorphism on the space $M_{a,b,\alpha}(A)$.

PROOF : S_a is clearly well defined and linear. For continuity we observe that

$$\sup_{R_+} \| \lambda_{a,b}(t)\, t^{k+1}\, D_t^k\, \phi(a.\, t) \|$$

$$= \sup_{R_+} \| \lambda_{a,\,b}\left(\frac{T}{a}\right) \left(\frac{T}{a}\right)^{k+1} a^k\, D_T^k\, \phi(T) \|,\ T = at$$

$$= C \sup_{R_+} \| \lambda_{a,b}(T)\, T^{k+1}\, D_T^k\, \phi(T) \|, \quad C \text{ is a constant}$$

$$= C\, i_{a,b,k}(\phi).$$

The inverse mapping S_a^{-1} is defined by $S_a^{-1} : \phi(t) \to \phi\left(\frac{t}{a}\right)$.

Theorem 4.2—If $\phi \in M(w, z)$, then, for each fixed t, $\phi(t\,x) \in M(W, z)$ as a function of x, where $x > 0$.

PROOF : Proof follows immediately from Theorem 4.1.

Theorem 4.3—If f is a member of $[M(w, z);\ A]$ and $\phi \in M(w, z)$, then $\phi \to \psi$ is a continuous linear mapping of $M(w, z)$ into $M(w, z;\ A)$, where

$$\psi(t) = < f(x), \phi(t\,x) >. \qquad ...(4.1)$$

PROOF :We first prove that $\psi(t)$ is an A-valued smooth function on $0 < t < \infty$, i. e. we want to prove

$$\psi^k(t) = < f(x), D_t^k\, \phi(tx) >. \qquad ...(4.2)$$

The above result is true for $k = 0$ by definition.

For $k = 1$, t fixed and $\Delta\, r \neq 0$ consider

$$\frac{\psi(t+\Delta t) - \psi(t)}{\Delta t} - < f(x), D_t\, \phi(t\,x) > = < f(x), \theta_{\Delta t}(x) > \quad ...(4.3)$$

where

$$\theta_{\Delta t}(x) = \frac{1}{\Delta t}[\phi\{(t+\Delta t)x\} - \phi(t\,x)] - D_t\,\phi(t\,x).$$

We now show that $\theta_{\Delta t}(x)$ converges in $M(w, z)$ to zero as $\Delta t \to 0$. Using Taylor's formula and treating Δt as independent variable, it can be easily proved that

$$|\lambda_{a,b}(x) \quad x^{p+1} \;\{\theta^{p}_{\Delta t}(x)\}| \to 0 \text{ as } \Delta t \to 0.$$

Thus result is established for $k = 1$. Now by induction the proof can be easily completed.

Next to prove $\psi(t) \in M(w, z; A)$, using the boundedness property of distributions we note that

$$\|\lambda_{a,b}(t)\; t^{k+1} \;\; D_t^k \;\phi(t)\|$$

$$= \|\lambda_{a,b}(t)\, t^{k+1}\; D_t^k \;< f(x), \phi(t\,x) > \|$$

$$\leqslant M \sup_{R_+} \;|\lambda_{a,b}(t)\, t^{k+1}\, \lambda_{a,b}(x)\, x^{p+1}\; D_x^p \;\; D_t^k\; \phi(t\,x)|$$

$$\leqslant M \operatorname{sum}_{R_+} |\lambda_{a,b}(t\,x)\,(t\,x)^{p+k+1}\, D^{p+k}\,\phi(t\,x)|$$

$$\leqslant M\, i_{a,b,p_+k}(\phi). \qquad \ldots(4.4)$$

Thus $\psi(t) \in M(w, z; A)$. Continuity of the mapping $\phi \to \psi$ also follows from (4.4).

Theorem 4.4—The mapping $f \to \psi$ defined by

$$\psi(t) = < f(x), \phi(\mathrm{t}\,x)>$$

is a linear mapping that is uniformly continuous with respect to S sets in $M(w, z)$.

PROOF : Let Ω be any S set in $M(w, z)$. We choose a and b such that $w < a < b < z$. Then Ω is a bounded set of $M_{a,b}$. Now $\psi^k(t) = <f(x), \;\phi^k(t\,x)>$ so that

$$\lambda_{a,b}(x)\, x^{m+1}\, D_x^m\; [\lambda_{a,b}(t)\; t^{k+1}\;\; D_t^k\, \phi(t\,x)]$$

$$= \lambda_{a,b}(t\,x)\,(t\,x)^{k+m+1}\; D^{k+m}\;\phi(t\,x). \qquad \ldots(4.5)$$

From (4.5), for k fixed, as ϕ traverses Ω and t traverses R_+, $\lambda_{a,b}\;(t)\, t^{k+1}\, D_t^k \;\; \phi(t\,x)$ as a function of x traverses a bounded set in Θ in $M_{a,b}$. This means that Θ is a S set in $M(w, z)$, and we have

$$\sup_{\phi \in \Omega} \sup_{t \in R_+} \|\lambda_{a,b}(t)\, t^{k+1}\, \psi^k(t)\|_A$$

$$= \sup_{\phi \in \Omega} \sup_{t \in R_+} \|\lambda_{a,b}(t)\, t^{k+1} < f(x),\ \phi^k(t\,x) > \|_A$$

$$= \sup_{\theta \in \Theta} \|< f, \theta > \|_A. \qquad \ldots(4.6)$$

Now if Λ is a neighbourhood of zero in $M(w, z; A)$, $(\Lambda) \cap M_{a,b}(A) \subset (\Lambda)$ is a neighbourhood of zero in $M_{a,b}(A)$. From (4.6), there exists a neighbourhood Ξ of zero in $[M(w, z); A]$ such that $\psi \in (\Lambda)$ for all $\phi \in \Omega$ and all $f \in \Xi$.

We now define Mellin type convolution.

Mellin Type Convolution—Let a and b be two real numbers with $a \leqslant b$. Mellin type convolution is an operation that assigns to each arbitrary choice of the pair $y \in [M(w, z; A): B]$ and $f \in [M(w, z); A]$ the product $y \vee f$ defined by

$$<y \vee f, \phi> = <y(t), <f(x), \phi(t\,x)>>$$
$$\phi \in M(w, z) \qquad \ldots(4.7)$$

Theorem 4.5—Assuming the validity of Theorems 4.2, 4.3 and 4.4. The operator $y \vee : f \to y \vee f$ is a continuous linear mapping of $[M(w, z), A]$ into $[M(w, z); B]$.

PROOF : Linearity is clear from (4.7). To prove continuity we have to prove that any neighbourhood Λ of zero in $[M(w, z); B]$ contain a neighbourhood Ξ of zero in $[M(w, z); A]$.

Let Φ be an arbitrary $s-$ set in $M(w, z)$. We have

$$\sigma_\Phi(y \vee f) = \sup_{\phi \in \Phi} \| <y \vee f, \phi> \|_B$$

$$= \sup_{\phi \in \Phi} \| <y, \psi> \|_B.$$

Because $y \in [M(w, z; A); B]$, to each neighbourhood Γ of zero in B there exists a neighbourhood Ω of zero in $M(w, z; A)$ such that y maps Ω into Γ. Now the proof can be completed by using the fact that $y \vee f$ is a composite mapping $\phi \to \psi \to <y, \psi>$.

We now prove in the following theorem that the operator $y \vee$ commutes with the scaling operator S_a.

Theorem 4.6—If $a > 0$, $y \in [D_+(A)\, B]$ and $f \in [E_+, A]$, then

$$S_a(y \vee f) = y(S_a f).$$

PROOF : By Theorem 4.1 $S_a : \phi(t) \to \phi(a\,t)$ is an automorphism on D_+ and hence

$$<S_a(y \vee f), \phi(t\,x)> = <(y \vee f), \frac{1}{a}\phi\left(\frac{t\,x}{a}\right)>$$

(*equation continued on p.* 500)

$$= <y(t) < f(x),\ \frac{1}{a}\phi\left(\frac{tx}{a}\right)\gg. \qquad \ldots(4.8)$$

Also

$$<y\,\nu\,(S_a f),\ \phi(tx)>$$

$$= <y(t) <S_a f(x),\ \phi(tx)\gg$$

$$= <y(t) <f(x),\ \frac{1}{a}\phi\left(\frac{tx}{a}\right)\gg. \qquad \ldots(4.9)$$

From (4.8) and (4.9) our theorem is proved.

Theorem 4.7—If $f \in [M_{a,b}\ (A), B]$ and $w \in D_+(A)$. Then, $w \to g$ is a continuous linear mapping of $D_+(A)$ into $E_+(B)$, where

$$g(x) = <f(t),\ \frac{1}{t}\ w\left(\frac{x}{t}\right)>.$$

PROOF : It is easy to prove that g is smooth and the mapping is linear. To prove continuity we observe that

$$\|g^k(x)\|_B = \|<f(t), D_x^k\left[\frac{1}{t}\ w\left(\frac{x}{t}\right)\right]>$$

$$\leqslant \max_{0 \leqslant 1 \leqslant r}\ \sup_{R_+}\ \|D_x^k\ D_t^1\ \frac{1}{t}\ w\left(\frac{x}{t}\right)\|.$$

From above continuity easily follows.

We call

$$(f\,\nu\,w)(x) = <f(t),\ \frac{1}{t}\,w\left(\frac{x}{t}\right)>.$$

The Mellin type regularization of f by w.

Theorem 4.8—If $y \in [D_+(A); B]$ and $My = Y(s)$ for $s \in \Omega_y$, if $f \in [D_+; A]$ and $Mf = F(s)$ for $s \in \Omega_f$, and if $\Omega_y \cap \Omega_f$ is not empty then $y\,\nu\,f$ exists in the sense of Mellin type. Convolution in $[M(w, z), B]$ where the interval (w, z) is the intersection of $\Omega_y \cap \Omega_f$ with the real axis. Moreover,

$$M(y\,\nu\,f) = Y(s)\ F(s).$$

PROOF : It is already proved in Theorem 4.5 that $y\,\nu\,f \in [M(w, z); B]$. Further because $t^{s-1} \in M(w, z)$ for each fixed s with $w < \text{Re}(s) < z$,

$$M(y\,\nu\,f) = <y(t), f(x), (tx)^{s-1}>$$

$$= <y(t), t^{s-1}> <f(x)\ x^{s-1}>$$

$$= Y(s) \cdot F(s).$$

Note that $Y(s)\ \ F(s)$ is B-valued function for any fixed $s \in \Omega_y \cap \Omega_f$.

5. Form Invariant Linear Filtering

Generalizing the result of Braccini and Gambardella[4] (p. 1613) we write the input output relationship for a linear one dimensional system as

$$g(x) = \langle f(t),\ w(t, x) \rangle \qquad \ldots(5.1)$$

where $f(t) \in [E(A), B]$, $w(t, x) \in E(A)$ and $g(x)$ is the output signal. When $A = B = \mathbb{C}$ the Banach space valued distribution f becomes scalar valued distribution. Further if $f(t)$ is a regular scalar valued distribution generated by locally integrable function f, we can write (5.1) as

$$g(x) = \int_{-\infty}^{\infty} f(t)\, w(t, x)\, dt. \qquad \ldots(5.2)$$

Further generalizing the definition of Braccini and Gambardella[4] we define the Banach space valued distributional form invariance property as below :

Let $g_a(x)$ be the output of the system (5.1) when the input $f(t)$ is replaced by $f(at)$, a being a positive real number. We say that the system is Banach space valued distributional form invariant if and only if

$$g_a(x) = \alpha\, g(\beta x) \qquad \ldots(5.3)$$

where α and β are real functions of a. From (5.1)

$$g_a(x) = \langle f(a\,t),\ w(t, x) \rangle$$

and

$$\alpha\, g(\beta\, x) = \alpha \langle f(t),\ w(t, \beta\, x) \rangle.$$

From (5.3) we get

$$\langle f(a\,t),\ w(t, x) \rangle = \alpha \langle f(t),\ w(t, \beta x) \rangle. \qquad \ldots(5.4)$$

Using the following property of distributions, namely

$$\langle f(a\,t), \phi(t) \rangle = \langle f(t),\ \frac{1}{a}\, \phi\left(\frac{t}{a}\right) \rangle$$

we get from (5.4)

$$\frac{1}{a} \langle f(t),\ w\left(\frac{t}{a}, x\right) \rangle = \alpha \langle f(t),\ w(t, \beta x) \rangle.$$

The above equation is true if and and only if

$$\frac{1}{a}\, w\left(\frac{t}{a}, x\right) = \alpha\, w(t, \beta x). \qquad \ldots(5.5)$$

General solution of this equation is (see Braccini and Gambardella[4], p. 1613).

$$w(t, x) = x^{-s}\ w_0(t/x^{\sigma}) \qquad \ldots(5.6)$$

$$\beta = a^{1/\sigma},\ \alpha = a^{(s-\sigma)/\sigma},\ x \neq 0$$

where δ and σ are real numbers. In (5.6) choice of σ affects the scale factor β of the output signal $\alpha\, g\, (\beta\, x)$, whereas the choice of δ (for any given value of σ) affects the amplitude factor α of the output $\alpha\, g\, (\beta\, x)$.

We now discuss some particular cases of (5.6) and relate the input output signals of the form invariant systems using Banach space valued distributional Mellin transform.

Case 1—Taking $\delta = \sigma = 1$, $\beta = a$, $\alpha = 1$ in (5.6), we get

$$w\,(t,\, x) = x^{-1}\, w_0\,(t-x),\; x \neq 0. \qquad \ldots(5.7)$$

After some simple manipulation (5.7) can be written equivalently as

$$w\,(t,\; x) = t^{-1}\, w_0\,(x/t),\;\; t \neq 0. \qquad \ldots(5.8)$$

Note that taking $\beta = a$ implies the same scaling factor in both the input $f\,(at)$ and the output $\alpha\, g\,(\beta\, x)$ of system (5.1). Further $\alpha = 1$ means no amplitude gain changes are undergone by the output $\alpha\, g\,(\beta x)$, when the input is linearly scaled.

Now using (5.8) we write input output relationship from (5.1) as

$$g\,(x) = \langle f\,(t),\;\, t^{-1}\, w_0\;(x/t) \rangle = (\, f\, v\, w_0). \qquad \ldots(5.9)$$

Note that here f is a Banach space valued distribution and not an ordinary integrable function as in Braccini and Gambardella.

Taking Mellin transform of both sides of (5.9), we get

$$G\,(s) = F\,(s).\; W_0\,(s). \qquad \ldots(5.10)$$

(5.10) is similar to relationship relating to the Laplace transforms of the output and the input of a shift invariant (time invariant) systems (see Zemanian[1]). The result is in agreement with result of Braccini and Gambardella[4] (p. 1618) when $f\,(t)$ is a Lebesgue integrable function.

To discuss next particular case of (5.6) we need

Theorem 5.1—The operator $P : M_{a-r,b-r,\alpha}\,(A) \rightarrow M_{a,b,\alpha}\;(A)$ defined by $P\;(\phi) = t^r\, \phi$ is an isomorphism from the space $M_{a,r,b-r,\alpha}\,(A)$ onto $M_{a,b,\alpha}\,(A)$ where r is a real number.

Proof : Observe that

$$\sup_{\mathbf{R}_+}\; \|\lambda_{a,b}\,(t)\;\, t^{k+1}\, D_t^k\;\, (t^r\, \phi)\|$$

$$= \sup_{\mathbf{R}_+}\; \|\lambda_{a,b}\,(t)\; t^{k+1}\; \sum_{p=0}^{k}\; C_p\;\, t^{r-p}\;\, D_t^{k-p}\;\;\, \phi\|$$

where C_p is some constant.

$$= \sup_{R_+} \| \sum_{p=0}^{k} C_p \lambda_{a-r,\, b-r} t^{k+1-p} D_t^{k-p} \phi \|$$

$$\leqslant \sum_{p=0}^{k} C_p \; i_{a-r, b-r, k-p} \; (\phi).$$

This proves the continuity of the map P. Linearity of the map is easily seen. The inverse mapping p^{-1} : $M_{a,b,\alpha}(A) \to M_{a-r,b-r,\alpha}(A)$ is defined by $p^{-1}(\phi) = t^{-r} \phi$. The linearity and continuity of p^{-1} can also be proved similarly. Thus P is an isomorphism.

Case II—Taking $w(t, x) = t^{\eta} \; x^{\mu} w_0 (x/t)$ where η and μ are arbitrary real numbers.

$$g(x) = < f(t),\; w(t, x) > \text{ as}$$

$$g(x) = < f(t), t^{-1} t^{\eta+1} x^{\mu} w_0(x/t) >$$
$$= [f(t) \; v \; t^{\eta+1} x^{\mu} w_0 (x/t)].$$

Using Theorem 5.1 and some simple properties of Mellin transform (see Snedonn[13], p. 270) we have

$$G(s) = F(s + \eta + \mu + 1) \; w_0 (s + \mu).$$

This result is generalization of the result obtained by Braccini and Gambardella[4] (p. 1618).

Concluding Remarks

In this paper we have applied Banach space valued distributional Mellin transform to 1-dimensional linear filters. In future we plan to extend these results to 2-dimensional linear filters. For this we will develop two dimensional Mellin transform and Laplace-Mellin (or Fourier-Mellin) transform. These transforms will be used to relate the input output signals. We will also prove that form invariance implies the operator has a convolution (Mellin type) representation.

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