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TORSIONAL VIBRATION OF A RANDOM ELASTIC CYLINDER

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We consider a random eigen-value problem arising from torsional vibrations of a cylinder with randomly varying density. The problem is transformed into a perturbed Fourier-Bessel equation and the perturbation technique is used to obtain its solution. An expression for variance of the eigen-value is also derived.

INTRODUCTION

We consider the equation

$$\mu \left\{ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right\} = \rho(r, \tau) \frac{\partial^2 u_\theta}{\partial t^2} \qquad \ldots(1)$$

where μ is the shear modulus, ρ the density and r, θ, z are the cylindrical polar co-ordinates. This equation arises from torsional vibrations of an elastic cylinder. The displacement of the z-axis is assumed to be zero so that

$$u_\theta = 0, \text{ when } r = 0 \text{ for all } z, t. \qquad \ldots(2)$$

Also the lateral surfaces of the cylinder are stress free. This gives rise to the boundary condition

$$\sigma_{r\theta} = \mu \left\{ \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right\} = 0 \text{ at } r = a$$
$$\text{for all } z \text{ and } t. \qquad \ldots(3)$$

This problem has been studied by Achenbach[1] when the cylinder is hollow and Heath and Wood[3] when the rigidity of the cylinder varies radially. In this paper we use perturbation technique to study this problem when the density is a function of r and a random parameter τ. This problem is of interest due to the fact that many new composite materials are effectively described as materials with random properties. In an earlier paper Zaman[8] has studied a random eigen-value problem arising from vibration of random elastic plates using the variational formulation. However, in this case we use perturbation method as the variational method does not seem to be applicable.

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Formulation of the Problem

We seek sinusoidal solutions of (1) and put

$$u_\theta = V(r) \exp [i(\omega t - \alpha z)] \quad \ldots(4)$$

where ω is the angular frequency and α is the reduced wave number. The equation of motion (1) and the boundary conditions (2) and (3) transform into

$$V''(r) + \frac{1}{r} V'(r) + \left[\frac{\rho(r, \tau)}{\mu} \omega^2 - \alpha^2 - \frac{1}{r^2} \right] V(r) = 0 \quad \ldots(5)$$

with

$$\left.\begin{aligned} V(0) &= 0 \\ V'(a) &= \frac{1}{a} V(a) \end{aligned}\right\} \quad \ldots(6)$$

where the time dependence $e^{i\omega t}$ is understood and omitted throughout. Assuming that the density ρ can be written as

$$\rho(r', \tau) = (\rho_0 + \epsilon \rho_1(r, \tau)) \quad \ldots(7)$$

where ρ_0 is the mean of ρ and ρ_1 is a random function with zero mean and ϵ is a small parameter. Equation (5) can thus be written as

$$V''(r) + \frac{1}{r} V'(r) + \left[\gamma - \epsilon \nu(r) - \frac{1}{r^2}\right] V(r) = 0 \quad \ldots(8)$$

where

$$\nu(r) = \frac{-\rho_0 \rho_1(r, \tau) \omega^2}{\mu}$$

and

$$\gamma = \frac{\rho_0 \omega^2}{\mu} - \alpha^2.$$

Equation (8) is a perturbation of Bessel's equation of order 1. The boundary conditions (6) and eqn. (8) form the perturbed random boundary value problem. Neither the perturbed nor the unperturbed differential expression is formally self-adjoint and both have a singularity at $r = 0$.

Solution of the Problem

We transform (8) into Liouville's normal form by putting

$$V(r) = \exp \left[-\tfrac{1}{2} \int^{r} \frac{1}{t} \, dt \right] u(r). \quad \ldots(9)$$

Equation (8) thus becomes

$$u''(r) + \left[\gamma - \frac{3}{4r^2} - \epsilon \nu(r)\right] u(r) = 0. \quad \ldots(10)$$

When $\epsilon = 0$, equation (10) reduces to the Fourier-Bessel equation with solutions

$r^{1/2} J_1 (kr)$, $r^{1/2} Y_1 (k r)$, where

$$k = \left(\frac{\rho_0 \omega^2}{\mu} - \alpha^2 \right)^{1/2}$$

which is real and positive whenever r is real and positive (Titchmarsh[6]; p. 81).

The transformation (9) transforms the boundary conditions (6) into

$$\left. \begin{aligned} u(0) &= 0 \text{ at } r = 0, \\ u'(a) &= \tfrac{3}{2} u(a) \text{ at } r = a. \end{aligned} \right\} \qquad \ldots(11)$$

The problem (10) with $\epsilon = 0$ is now self-adjoint with one singular end-point at $r = 0$ where it is the limit point case[6]. It has a countable number of real simple eigen-values $\gamma_0, \gamma_1, \gamma_2, \ldots$ where $\gamma_0 = 0$ and $J_1 (k_n a) = 0$ for $n = 1, 2, \ldots$.

The associated eigen-functions are

$$\left. \begin{aligned} u_0 (r) &= A_0 r^{3/2} \\ u_n (r) &= A_n r^{1/2} J_1 (k_n r), \; n > 1. \end{aligned} \right\} \qquad \ldots(12)$$

With a suitable choice of A_n, $\{u_n\}_{n=0}^{\infty}$ is an orthonormal set in $L^2 (0, a)$.

We now write

$$U_n (r) = u_n (r) + \epsilon\, u_n^{(1)}(r) + \epsilon^2 \; u_n^{(2)} (r) + \ldots \qquad \ldots(13)$$

$$A_n = \gamma_n + \epsilon \gamma_n^{(1)} + \epsilon \gamma_n^{2(2)} \ldots \qquad \ldots(14)$$

where $u_n (r)$ and γ_n are the nth eigen-function and eigen-value of the unperturbed problem and are given by (12). Following Titchmarsh[6], we write $u_n^{(1)}$, $u_n^{(2)}$ as expansions in terms of the eigen-functions of the unperturbed problem as

$$u_n^{(1)} (r) = \sum_{p=0}^{\infty} \alpha_{np}\, u_p (r)$$

$$u_n^{(2)} (r) = \sum_{p=0}^{\infty} \beta_{np}\, u_p (r). \qquad \ldots(15)$$

Using (13), (14) and (15) in the perturbed equation (10) and equating the coefficient of ϵ to zero, we get

$$u_n^{(1)''} (r) + \left(\gamma_n - \frac{3}{4r^2} \right) u_n^{(1)} (r) + (\gamma_n^{(1)} - \nu(r)) \times u_n (r) = 0. \qquad \ldots(16)$$

We multiply equation (16) by $u_n(r)$ and integrate using the orthonormality of $u_n(r)$' s to get

$$\gamma_n^{(1)} = \frac{-\rho_0\,\omega^2}{\mu}\int_0^a \rho_1(r,\tau)\,u_n^2\,(r)\,dr \qquad ...(17)$$

where we have substituted value of $v(r)$ introduced earlier. In a similar way, we multiply (16) by $u_m(r)$ and integrate to get

$$\alpha_{nm} = \frac{-1}{\gamma_n-\gamma_m}\left[\frac{\rho_0\,\omega^2}{\mu}\int_0^a \rho_1(r,\tau)\,u_n(r)\right.$$

$$\left. u_m\,(r)\,dr + \int_0^a \gamma_n^{(1)}\,u_n(r)\,u_m(r)\,dr\right] \qquad ...(18)$$

where $\gamma_n^{(1)}$ is given by (17).

Estimate on the Variance

We note that $\gamma_n^{(1)}$ is a weakly correlated random process (Boyce[2]). It is of the form

$$\gamma_n^{(1)} = \int_0^a f\,(r)\,P\,(r,\tau)\,dr \qquad ...(19)$$

where

$$f(r) = \frac{\rho_0\,\omega^2}{\mu}u_n^2\,(r)$$

and

$$P\,(r,\tau) = \rho_1\,(r,\tau). \qquad ...(20)$$

The mean square of such a process is given by

$$<\gamma_n^{(1)2}> = \int_0^a\int_0^a K\,(r_1,r_2)\,f\,(r_1)\;f\,(r_2)\,dr_1\,dr_2 \qquad ...(21)$$

where $K(r_1, r_2)$ is the correlation function for the random quantity $\rho_1(r, \tau)$. There are well documented experimental methods for determination of the correlation function appearing in (21) (Corson[4] Miller[5]). Zaman[7] has given an elementary method to derive the correlation function for a statistically homogeneous two phase composite. For two phase composites, $K(r_1, r_2)$ is given by

$$K(r_1, r_2) = \rho_{1,1}^2 P_{11}(A, B) + \rho_{1,1}\rho_{1,2} P_{12}(A,B)$$

$$+ \rho_{1,2}\rho_{1,1} P_{21}(A, B) + \rho_{2,2}^2 P_{22}(A, B) \quad \ldots(22)$$

where the property ρ_1 is related at two points A and B lying anywhere in the material. $P_{ij}(A, B)$ is the probability that point A is in phase i and point B is in the phase j of the material and $\rho_{1,j}$ is the value of ρ_1 in the jth material for $j = 1, 2$. Using arguments based upon elementary probability theory we find[7]

$$K(r_1, r_2) = < \rho_1^2 > \exp\left\{ \frac{-3S \mid r_1 - r_2 \mid}{8c(1 - c)V} \right\} \quad \ldots(23)$$

where c is the volume concentration of the material forming phase 1, S is its surface area and V is the total volume. Thus we can assume the following form

$$K(r_1, r_2) = < \rho_1^2 > \exp\{-\alpha \mid r_1 - r_2 \mid\} \quad \ldots(24)$$

where α is a parameter depending upon nature of the composite material.

References

1. J. D. Achenbach, *Rev. Roum. Sci. Tech., series de Mecanique Applique*, 1966.
2. W. E. Boyce *Random Eigenvalue Problems : Probabilistic Methods in Applied Mathematics* (ed.: A. T. Bharucha Reid), Academic Press, 1968.
3. J. D. Heath and A. D. Wood, *Proceedings of the Conference on Ordinary and Partial Differential Equations* (ed.: B. D. Sleeman and I.M. Michael). Springer Verlag, Vol. **415** (1974), 383–87.
4. P. B. Corson *J. Appl. Phys.* **45** (1974), 3159–70.
3. M. N. Miller, *J. Math. Phys.* **10** (1969), 1988–2019.
6. E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second Order Differential Equations*. Oxford University Press, 1962.
7. F. D. Zaman, *J. Nat. Sci. Math.* **19** (1979) 63–70.
8. F. D. Zaman, *J. Nat. Sci. Math.* **29** (1986), 35–44.