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A FIXED POINT THEOREM FOR GENERALIZED CONTRACTION MAP

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In this paper we prove a fixed point theorem for a generalized contraction map introduced by Altman and then derive a few known results as corollaries.

Altman[2] proved the following interesting theorem : Let x be a complete metric space and $f : x \rightarrow x$ a generalized contraction, i.e.,

$$d(fx, fy) \leqslant Q(d(x, y)) \text{ for all } x, y \in X,$$

where Q satisfies the following :

(a) $0 < Q(t) < t$, for all $t \in (0, t_1]$,

(b) $g(t) = t/(t - Q(t))$ is nonincreasing,

(c) $\int_0^{t_1} g(t)\, dt < \infty$

and

(d) Q is nondecreasing.

Then f has a unique fixed point (see also Altman[1]).

Recently Watson *et al.*[6] pointed out that the fixed point is not necessarily unique under conditions (a), (b) (c) and (d). Carbone and Singh[3] gave a suitable example showing that the fixed point is, indeed, not unique.

Watson *et al.*[6] proved a theorem for a pair of mappings showing that $Fx = Gx$ has a unique solution under a set of conditions, where F is a generalized contraction and G is an expansive map. Their theorem improves a result due to Norris and Sehgal[4].

Our aim is to prove the following theorem and to derive a few known results as corollaries.

Theorem 1—Let X be a complete metric space and let $f, h : X \to X$ be continuous functions such that

$$d(hx, hy) \leqslant Q(m(x, y)) \text{ for } x, y, \in X$$

where

$$m(x, y) = \max\left\{d(fx, fy), d(fx, hx), d(fy, hy), \frac{d(fx,hy) + d(fy,hx)}{2}\right\}.$$

Also suppose

(i) f and h are weakly commuting, i.e.

$$d(hfx, fhx) \leqslant d(fx, hx), \text{ and}$$

(ii) $h(X) \subset f(X)$.

Then f and h have a unique common fixed point. (i.e., there exists $x_0 \in X$ such that $fx_0 = x_0 = hx_0$).

In this case Q satisfies the following :

Q is a real-valued function such that

(a) $0 < Q(y) < y$ for $y > 0$, and $Q(0) = 0$,

(b) $g(y) = y/(y - Q(y))$ is nonincreasing on $(0, \infty)$,

(c) $\int_0^{y_1} g(y)\, dy < \infty$ for each $y_1 > 0$,

and

(d) $Q(y)$ is nondecreasing.

PROOF : Suppose x and y are distinct common fixed points of f and h. Then $m(x, y) > 0$, since $fx \neq hy$. Hence,

$$d(hx, hy) \leqslant Q(m(x, y))$$

$$< \max\{d(fx, fy), 0, 0, d(fx, fy)\},$$

a contradiction.

To prove the existence, take x_0 in X and set $t_1 = d(hx_0, fx_0)$. Suppose $t_1 = 0$. Then

$$d(hhx_0, hx_0) \leqslant Q(m(hx_0, x_0))$$

where

$$m(hx_0, x_0) = \max \left\{ d(fhx_0, fx_0), d(fhx_0, hhx_0), d(fx_0, hx_0), \frac{d(fhx_0, hx_0) + d(fx_0, hhx_0)}{2} \right\}.$$

Since f and h are weakly commuting and $fx_0 = hx_0$,

we have

$$d(fhx_0, hhx_0) = 0.$$

Hence

$$m(hx_0, x_0) = d(hhx_0, hx_0).$$

Note that $m(hx_0, x_0)$ must be zero, otherwise $m(hx_0, x_0) > 0$ would imply

$$d(hhx_0, hx_0) \leqslant Q(m(hx_0, x_0)) < d(hhx_0, hx_0)$$

a contradiction.

Thus $m(hx_0, x_0) = 0$, i.e., hx_0 is a fixed point of h.

But then

$$ffx_0 = fhx_0 = hhx_0 = hx_0 = fx_0$$

i.e.,

$$fx_0 = hx_0 \text{ is a fixed point of } f.$$

We may assume, now that $t_1 > 0$. Since $h(X) \subset f(X)$ there exists an $x_1 \in X$ with $fx_1 = hx_0$. In general, define $\{x_n\} \subset X$ so that $fx_n = hx_{n-1}, n \geqslant 1$.

Without loss of generality we may assume that $fx_n \neq hx_n$ for each n. For if $fx_n = hx_n$ for some n, then a repeat of the above argument, with x_0 replaced by x_n, yields fx_n as a common fixed point of f and h.

Define $\{t_n\}$ by $t_{n+1} = Q(t_n)$, with $t_1 = d(hx_0, fx_0)$. It then follows by assumption a) of Theorem 1 that

(i) $0 < t_{n+1} \leqslant t_n \leqslant t_1, n \geqslant 1$. Moreover, by hypotheses (b) and (c), the series $\sum_{n \geqslant 1} t_n$ converges (see Altman[1]). Furthermore, by induction on $n \in N$, we have

(ii) $d(hx_n, hx_{n-1}) \leqslant t_{n+1}, n \geqslant 1$.

Indeed, for $n = 1$,

$$d(hx_1, hx_0) \leqslant Q(m(x_1, x_0))$$

where

$$m(x_1, x_0) = \max\left\{d(fx_1, fx_0), d(fx_1, hx_1), d(fx_0, hx_0), \frac{d(fx_1, hx_0) + d(fx_0, hx_1)}{2}\right\}$$

$$= \max\left\{d(hx_0, fx_0), d(hx_0, hx_1), d(fx_0, hx_0), \frac{d(fx_0, hx_1)}{2}\right\}$$

$$= \max\{d(hx_0, fx_0), d(hx_0, hx_1)\} > 0.$$

Now, if $m(x_1, x_0) = d(hx_0, hx_1)$, then

$$d(hx_1, hx_0) \leqslant Q(m(x_1, x_0)) < d(hx_0, hx_1)$$

a contradiction.

Then

$$m(x_1, x_0) = d(hx_0, fx_0) = t_1.$$

Thus (ii) is proved for $n = 1$.

Assume now that (ii) holds for some $n > 1$. Then

$$d(hx_{n+1}, hx_n) \leqslant Q(m(x_{n+1}, x_n)),$$

where

$$m(x_{n+1}, x_n) = \max\left\{d(fx_{n+1}, fx_n), d(fx_{n+1}, hx_{n+1}), d(fx_n, hx_n), \frac{d(fx_{n+1}, hx_n) + d(fx_n, hx_{n+1})}{2}\right\}$$

$$= \max\{d(hx_{n+1}, hx_n), d(hx_n, hx_{n-1})\}.$$

Note that by the assumption $fx_n \neq hx_n$ for all n, $m(x_{n+1}, x_n) > 0$ for all n. If $m(x_{n+1}, x_n) = d(hx_{n+1}, hx_n)$, then we get

$$d(hx_{n+1}, hx_n) \leqslant Q(m(x_{n+1}, x_n)) < d(hx_{n+1}, hx_n), \text{ a contradiction.}$$

Therefore,

$$m(x_{n+1}, x_n) = d(hx_n, hx_{n-1})$$

and

$$d(hx_{n+1}, hx_n) \leqslant Q(d(hx_n, hx_{n-1}))$$

$$\leqslant Q(t_{n+1}) = t_{n+2}.$$

Clearly $\{hx_n\}$ is a Cauchy sequence. In fact, if m and n are natural numbers with $m \leqslant n$, then

$$d(hx_m, hx_n) \leqslant \sum_{i=m}^{n-1} d(hx_i, hx_{i+1}) < \sum_{i=m}^{n-1} t_{i+2}.$$

The convergence of $\sum_{n\geqslant 1} t_n$ implies that $\{hx_n\}$ is a Cauchy sequence, hence converges to a point $y \in X$. Since $hx_n = fx_{n+1}$, $\{fx_n)$ also converges to y. Since f is continuous we get $fhx_n \to fy$. But f and h weakly commute. Hence we get $d(hfx_n, fy) \leqslant d(hfx_n, fhx_n) + d(fhx_n, fy)$, and $hfx_n \to fy$.

Since h is also continuous, $hfx_n \to hy$, so $hy = fy$.

Then, a repeat of the argument at the beginning of the proof with x_0 replaced by y, yields $hy = fy$ as a common fixed point of f and h.

The following results follow as Corollaries :

Corollary 1—If we replace weakly commuting by the commuting property i.e. $fhx = hfx$ for all $x \in X$, in Theorem 1, then f and h have a unique common fixed point. Note : Recall that commuting maps are weakly commuting, but not conversely (see Sessa[5]).

Corollary 2—If $m(x, y)$ is replaced by $d(fx, fy)$ in Theorem 1, then f and h have a unique common fixed point.

Corollary 3—We get a result due to Carbone and Singh[3] by putting $d(fx, fy)$ for $m(x, y)$ and commuting for weakly commuting in Theorem 1.

Corollary 4—In Corollary 3, if we put $f = I$, the identity function, then we get a theorem of Watson *et al.*[6].

Theorem 1 can be used to find the solution of an operator equation of the form $hx = Gx$, under suitable conditions on G.

We state the following given in Watson *et al.*[6].

Theorem 2—Let $h, G : X \to X$ be such that

(i) h is as in Theorem 1 with $f = I$, and $m(x, y) = d(fx, fy)$,

(ii) $d(Gx, Gy) \geqslant d(x, y)$ for all $x, y \in X$ and

(iii) $h(X) \subseteq G(X)$.

Then $hx = Gx$ has a unique solution z and for every

$$x_0 \in X, \lim_{n\to\infty} (G^{-1} h)^n x_0 = z.$$

In this case $G^{-1} h$ satisfies the conditions of Corollary 4 (see Watson *et al.*[6] for details).

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