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SEQUENCES OF MAPPINGS CONVERGING TO A CONTRACTION MAPPING

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We consider sequences of mappings $\{T_n\}$ from a metric space X into itself, which converge to a contraction mapping T. We define $T^{n+1} = T_{n+1} T^n$, $T^1 = T_1$ and we investigate the convergence of the sequence $\{T^n x\}$ for any point x in X.

1. INTRODUCTION

Nadler[1] investigated metric spaces (X, d) and sequences of functions $\{T_n\}$, $T_n: X \to X$, which converge on X to a contraction mapping T with fixed point a. He supposed that the functions $\{T_n\}$ have at least one fixed point for each $n = 1, 2, 3, \ldots$ and proved the convergence of the sequence of these fixed points to the fixed point a of T.

In this article we consider such sequences of functions, we define $T^{n+1} = T_{n+1}T^n T^1 = T_1$ and study the convergence of the sequence $\{T^n x\}$ for a point x in X.

2. CONVERGENCE OF THE SEQUENCE IN A METRIC SPACE

If $\{T_n\}$ is a sequence of mappings from a space X into itself then for every positive integer n, we define $T^{n+1} = T_{n+1} T^n$ with $T^1 = T_1$. If $n > m$ and n is a positive integer, by T_m^n we mean: $T_m^n = T_n T_{n-1} T_{n-2} \ldots T_m$.

Theorem 1—Let (X, d) be a metric space, let $\{T_n\}$ be a sequence of functions of X into itself and let $T: X \to X$ be a contraction mapping with fixed point a. If the sequence $\{T_n\}$ converges uniformly to T, then the sequence $\{T^n x\}$ converges to a for all $x \in X$.

PROOF : The proof is given in three steps.

Suppose k is the Lipschitz constant of T and choose $k_1 \in \mathbb{R}$ such that $0 < k < k_1 < 1$.

Step I—We shall show that for any $\epsilon > 0$ there exists an $m_0 \in \mathbb{N}$ such that $n \geqslant m_0$ gives :

$$d(T^n a, T^n x) < \epsilon \qquad \ldots(2.1)$$

for every $x \in X$.

Let $\epsilon > 0$ be given. Then for $\epsilon' = (k_1 - k)\epsilon > 0$ there exists an $i_0 \in \mathbb{N}$ such that for every $i \geqslant i_0$ we have :

$$d(T_i x, Tx) < \frac{\epsilon'}{2} \text{ for all } x \in X.$$

So for $i \geqslant i_0$, if $d(T^{i-1} a, T^{i-1} x) \geqslant \dfrac{\epsilon'}{k_1 - k} = \epsilon$ we have :

$$\begin{aligned} d(T^i a, T^i x) &= d(T_i T^{i-1} a, T_i T^{i-1} x) \leqslant d(T_i T^{i-1} a, TT^{i-1} a) \\ &\quad + d(TT^{i-1} a, TT^{i-1} x) + d(TT^{i-1} x, T_i T^{i-1} x) \\ &< \frac{\epsilon'}{2} + kd(T^{i-1} a, T^{i-1} x) + \frac{\epsilon'}{2} \leqslant k_1 d(T^{i-1} a, T^{i-1} x) \end{aligned}$$

and if $d(T^{i-1} a, T^{i-1} x) < \dfrac{\epsilon'}{k_1 - k}$ it follows

$$d(T^i a, T^i x) < \frac{\epsilon'}{2} + kd(T^{i-1} a, T^{i-1} x) + \frac{\epsilon'}{2} \leqslant \epsilon' + \frac{k\epsilon'}{k_1 - k} < \epsilon.$$

Thus in any case there exists an $m_0 \in \mathbb{N}$ such that for $n \geqslant m_0$

$$\begin{aligned} d(T^n x, T^n a) &= d(T^n_{i_0} T^{i_0 - 1} x, T^n_{i_0} T^{i_0 - 1} a) \\ &< \max\{\epsilon, k_1^{n - i_0 + 1} d(T^{i_0 - 1} x, T^{i_0 - 1} a)\} = \epsilon. \end{aligned}$$

Step II—We will now prove that for any $\epsilon > 0$ there exists an n_0 such that if $n \geqslant n_0$

$$d(T^n a, a) < \epsilon. \qquad \ldots(2.2)$$

Choose $\epsilon > 0$ and $\epsilon' = (k_1 - k)\epsilon$. Then there exists an $i_0 \in \mathbb{N}$ such that for every $i \geqslant i_0$ it follows : $d(T_i x, Tx) < \epsilon'$ for all $x \in X$. So if $d(T^{i-1} a, a) > \dfrac{\epsilon'}{k_1 - k} = \epsilon$ we have

$$\begin{aligned} d(T^i a, a) &= d(T^i a, Ta) = d(T_i T^{i-1} a, Ta) \leqslant d(T_i T^{i-1} a, TT^{i-1} a) \\ &\quad + d(TT^{i-1} a, Ta) < \epsilon' + kd(T^{i-1} a, a) \leqslant k_1 d(T^{i-1} a, a) \end{aligned}$$

and if

$$d(T^{i-1} a, a) < \frac{\epsilon'}{k_1 - k}$$

then

$$d(T^i a, a) < \epsilon' + kd(T^{i-1} a, a) < \frac{k_1 \epsilon'}{k_1 - k} < \epsilon.$$

So finally, in any case, there exists an $m_0 \in \mathbb{N}$ such that for $n > m_0$ it follows :

$$d(T^n a, a) < \max\{\epsilon, k_1^{n - i_0 + 1} d(T^{i_0 - 1} a, a)\} = \epsilon.$$

Step III—Now by (2.1) and (2.2) if for $\epsilon > 0$ we choose $N = \max\{n_0, m_0\}$, then for $n \geqslant N$ we have :

$$d(T^n x, a.) \leqslant d(T^n x, T^n a) + d(T^n a, a) < \epsilon + \epsilon$$

and the theorem is proved.

The following example and Example 2 of the next section show that we can not omit from the assumptions of Theorem 1 either that T is a contraction mapping or that the sequence $\{T_n\}$ converges uniformly to T.

Example 1—$X = \mathrm{R}$, $T_n x = 2x - 3 - 1/n$, $Tx = 2x - 3$, and the distance function d is the ordinary euclidean distance on the line. Thus $\{T_n\}$ converges uniformly to T but $\lim_{n\to\infty} T^n x = +\infty$ if $x > 4$.

3. Convergence of the Sequence in a Locally Compact Metric Space

Theorem 2—Let (X, d) be a locally compact metric space, let $\{T_n\}$ be a sequence of contraction mappings of X into itself and let $T : X \to X$ be a contraction mapping with fixed point a. If the sequence $\{T_n\}$ converges pointwise to T then there exists an $r > 0$ and a positive integer m_0 such that $\lim_{n\to\infty} T^n_m\ x = a$ for every $x \in K(a, r) \equiv \{x : d(a, x) \leqslant r\}$, whenever $m \geqslant m_0$.

Proof : Choose $r > 0$ such that the set $K(a, r)$ is a compact subset of X. From the equicontinuity property it follows that the sequence $\{T_n\}$ is uniformly convergent on $K(a, r)$. We choose m_0 such that if $n \geqslant m_0$ then

$$d(T_n x, Tx) < (1 - k) r$$

for all $x \in K(a, r)$, where $k < 1$ is the Lipschitz constant for T. Then if $n \geqslant m_0$ and $x \in K(a, r)$ we have :

$$d(T_n x, a) \leqslant d(T_n x, Tx) + d(Tx, a) \leqslant (1 - k) r + kr = r$$

and thus if $n \geqslant m_0$, T_n maps $K(a, r)$ into itself. Now we take $k_1 \in \mathrm{R}$ with $0 < k < k_1 < 1$, and let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$, $N > m_0$ such that $n > N$ implies

$$d(T_n x, Tx) < \epsilon' = \epsilon(k_1 - k)$$

for all $x \in K(a, r)$, so that for these x and for $i - 1 \geqslant N > m \geqslant m_0$ if $d(T^{i-1}_m x, a) \geqslant \dfrac{\epsilon'}{k_1 - k}$ we have :

$$d(T^i_m\ x, a) \leqslant d(T_i\ T^{i-1}_m\ x, TT^{i-1}_m\ x) + d(TT^{i-1}_m\ x, Ta)$$

$$< \epsilon' + k\, d(T^{i-1}_m\ x, a) \leqslant k_1\, d(T^{i-1}_m\ x, a)$$

and if $d\ (T_m^{i-1}\ x, a) < \dfrac{\epsilon'}{k_1-k}$ it follows

$$d\,(T_m^i\ x, a) < \epsilon' + k\ \frac{\epsilon'}{k_1 - k} < \epsilon.$$

Thus in both cases there exists an $n_0 \in \mathcal{N}$, $n_0 > N$, such that $n \geqslant n_0$ gives

$$d\ (T_m^n\ x, a) < \max\{\epsilon,\ k_1^{n-N}\ d\ (T_m^N\ x, a)\} = \epsilon.$$

We now give an example which shows that in non-locally compact spaces a sequence of contraction mappings $\{T_n\}$ may converge pointwise to a contraction mapping T with fixed point a, but for every $r > 0$ and every positive integer m_0, we can find an $x \in K(a, r)$ such that the sequence $\{T_{m_0}^n\ x\}$ does not converge to the point a. We proceed as in Nadler[1].

Example 2—Let X be an infinite dimensional separable or reflexive Banach space. Let X^* be the first conjugate of X and let $T = \{f \in X^* : \|f\| \leqslant 1\}$. Then T is weak* sequentially compact. Since X is infinite dimensional, there is a sequence $\{g_k\}$ of linear functionals in T which has no norm convergent subsequence. Let $\{g_{k_i}\}$ be a weak* convergent subsequence of $\{g_k\}$ and let g be the weak* limit of $\{g_{k_i}\}$. For each $i = 1, 2, 3, \ldots$ let

$$f_i = \frac{g_{k_i} - g}{\|g_{k_i} - g\|}.$$

The sequence $\{f_i\}$ is weak* convergent to the zero linear functional and $\|f_i\| = 1$ for all $i = 1, 2, 3, \ldots$

For each $i = 2, 3, \ldots$ let $a_i \in X$ such that $\|a_i\| = 1$ and

$$|f_i\ (a_i)| > \left(1 - \frac{1}{i^2}\right) \Big/ \left(1 - \frac{1}{i^3}\right)$$

and define $T_i : X \to X$ by

$$T_i\ x = \left(1 - \frac{1}{i^3}\right) f_i\ (x)\ a_{i+1}$$

for all $x \in X$. It is easily seen that $\{T_i\}$ converges point-wise to the zero mapping. Since

$$\|T_i\ x - T_i\ y\| = \left(1 - \frac{1}{i^3}\right) |f_i\ (x) - f_i\ (y)|\ \|a_{i+1}\| \leqslant \left(1 - \frac{1}{i^3}\right)$$

$$\|x - y\|$$

for all x and y in X, T_i is a contraction mapping for each $i = 1, 2, 3, \ldots$ and we have

$$T_m^n\ x = \left(1 - \frac{1}{n^3}\right)\left(1 - \frac{1}{(n-1)^3}\right)\text{`}\ldots\text{'}\left(1 - \frac{1}{m^3}\right) f_m\ (x) f_{m+1}$$

$$\times\ (a_{m+1})\ \text{`}\ldots\text{'}\ f_n\ (a_n)\ a_{n+1}$$

with

$$\|T_m^n\ x\| > \prod_{i=m+1}^{n}\left(1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{m^3}\right)|\ f_m(x)\ | \geqslant \tfrac{1}{2}\left(1 - \frac{1}{m^3}\right)$$

$$\times\ |\ f_m\ (x)\ |\ .$$

We conclude with an analogue of Theorem 3 of Nadler[1] which characterizes finite dimensional spaces.

Theorem 3—A separable or reflexive Banach space X is finite dimensional if and only if whenever a sequence of contraction mappings of X into X converges pointwise to a contraction mapping T with fixed point a, then there exists an $m_0 \in N$ and an $r > 0$ such that $\lim_{n\to\infty} T_m^n\ x = a$ for all $x \in K(a, r)$ whenever $m \geqslant m_0$.

PROOF : The condition is obviously sufficient, since every finite dimensional Banach space is locally compact and thus Theorem 2 applies. The converse assertion follows from Example 2.

REFERENCE

1. S. B. Nadler (Jr), *Pacific J. Math.* **27** (1968) 579-85.