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NOETHERIAN REGULAR RINGS

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In this paper, we obtain various characterizations of Noetherian regular rings.

1. INTRODUCTION

It is well known that in a commutative ring with identity, every radical ideal can be represented as an intersection of prime ideals. In general, the representation is not unique. Here we observe that this is true in the case of Noetherian regular rings (Throughout in this paper "regular" means "Von Neumann regular"). We characterize Noetherian regular rings by means of saturated sets. Using this characterization we also prove that a commutative ring with identity is Noetherian regular if and only if it is semiprime in which every non unit is a zero-divisor and the zero ideal is a product of a finite number of principal ideals generated by semiprimary elements.

Throughout R denotes a commutative ring with identity. We now recall some definitions.

A nonempty subset S in R is said to be saturated if for any $x, y \in R$, $x, y \in S$ if and only if $xy \in S$. A proper saturated set S is said to be maximal if S is not contained in any proper saturated set of R.

It is easy to see that maximal saturated sets always exist in R.

Let $S(R)$ denote the set of all saturated sets of R. For any collection of S_α's in $S(R)$, define $VS_\alpha = \{x \in R \mid xy = f_{\alpha_1} f_{\alpha_2} \dots f_{\alpha_n}$ for some $y \in R$ and $f_{\alpha_i} \in S_{\alpha_i}\}$. Then $(S(R), V, \cap)$ is a lattice with 'V' as the supremum and set intersection '$\cap$' as the infimum. For any $a \in R$, the principal saturated set generated by 'a' is the intersection of all saturated sets containing 'a' and is denoted by $[a)$. Infact $[a) = \{x \in R \mid xy = a^n$ for some $y \in R$ and $n \in Z^+ \cup \{0\}\}$. It can be easily seen that $[a)$ is the smallest saturated set containing a.

An ideal A of R is said to be a radical ideal if $A = \sqrt{A} = \{x \in R \mid x^n \in A$ for some $n \in Z^+\}$. All ideals are assumed to be proper. For any $a \in R$, the principal ideal generated by a is denoted by (a); $P(R)$ denotes the set of all prime ideals of R.

2. Noetherian Regular Rings

We obtain various characterizations of Noetherian regular rings. We shall begin with the following lemma.

Lemma 1—Let R be semiprime and let F be a maximal saturated set in R. If $\sum_{i=1}^{n} a_i \in F$ then $a_i \in F$ for some i.

Proof : Suppose $a_i \notin F$ for all $i = 1, 2, \ldots, n$. As F is a maximal saturated set, we get $a_i f_i = 0$ for some $f_i \in F$, $1 \leqslant i \leqslant n$. Therefore $(\prod_{1=i}^{n} f_i) \cdot a_i = 0$ for $i = 1, 2, \ldots, n$ and so $(\prod_{i=1}^{n} f_i)(\sum_{i=1}^{n} a_i) = 0$. Thus $0 \in F$, a contradiction. Hence $a_i \in F$ for some i.

Remark 1 : The above lemma shows that the complement of any maximal saturated set in a semiprime ring is always a prime ideal.

Theorem 1—R is regular if and only if R is semiprime satisfying any one of the following two conditions :

(i) the complement of every maximal saturated set in R is a maximal ideal;
(ii) the complement of every maximal ideal in R is a maximal saturated set.

Proof : It can be easily seen that the conditions (i) and (ii) are equivalent in a semiprime ring R. If R is regular then R is semiprime and every prime ideal in R is a maximal ideal and hence by the above remark R satisfies condition (i). Now we assume that R is a semiprime ring satisfying condition (i). Let $x \in R$ and $T(x) = \{y \in R \mid xy = 0\}$, $D(x) = \{y \in R \mid (x) + (y) = R\}$. If $T(x) \cap D(x) \neq \phi$ then we are through. Suppose $T(x) \cap D(x) = \phi$. It is easy to verify that $T(x)$ is an ideal and that $D(x)$ is a saturated subset of R. Let $\mathcal{S} = \{S \in S(R) \mid D(x) \subseteq S, S \cap T(x) = \phi\}$. Partially order $\mathcal{S}$ by set-inclusion. By Zorn's lemma, $\mathcal{S}$ has a maximal element, say F. If $x \notin F$ then $(FV[x]) \cap T(x) \neq \phi$. Hence there is an element $y \in T(x)$ such that $y \in FV[x]$. As $xy = 0$ and $y \in FV[x]$ we have $fx^n = 0$ for some $f \in F$ so that $F \cap T(x) \neq \phi$ which is impossible. Therefore $x \in F$. Now we show that F is a maximal saturated set. Let $F \subset G$ for some saturated set G of R. Choose $z \in G - F$. Then $(FV[z]) \cap T(x) \neq \phi$ and so there exists an element $y \in T(x)$ such that $y \in FV[z]$. Since R is semiprime it now follows that $fzx = 0$ for some $f \in F$. Clearly $fx \in F \subset G$ and hence $0 = fzx \in G$. Consequently $G = R$. This shows that F is a maximal saturated set. Now by condition (i), $R - F$ is a maximal ideal and $x \notin R - F$, so $(R - F) + (x) = R$: i.e., $(y) + (x) = R$ for some $y \in R - F$, so $y \in D(x)$ which is impossible since $D(x) \subseteq F$. Hence $T(x) \cap D(x) \neq \phi$. Thus every principal ideal is a direct summand and hence R is regular.

Theorem 2—R is Noetherian regular if and only if R is semiprime and (*) every radical ideal has a unique representation as an intersection of prime ideals.

PROOF : Suppose R is Noetherian regular. Then R is semiprime and R contains only a finite number of prime ideals. Let I be an ideal in R. Then $I = \bigcap_{i=1}^{n} P_i = \prod_{i=1}^{n} P_i$, where P_i's are prime ideals in R. If $I = \bigcap_{i=1}^{m} Q_i = \prod_{i=1}^{m} Q_i$ where Q_j's are prime ideals in R, then it can be easily seen that each $P_i = Q_j$ for some j and each $Q_j = P_i$ for some i. This shows that the representation is unique.

Conversely assume that R is semiprime satisfying condition (*). First we show that every radical ideal is principal. Let I be a radical ideal. Put $J = \cap \{P \in P(R) \mid I \nsubseteq P\}$. Since R is semiprime, clearly $IJ = \cap \{P \mid P \in P(R)\} = (0)$ so that $I \cap J = (0)$. We show that $I + J = R$. If $I + J \neq R$ then $I + J \subseteq P_0$ for some prime ideal P_0 of R. Then $J = \cap \{P \mid I \nsubseteq P\} = (\cap \{P \mid I \nsubseteq P\}) \cap P_0$ so that J has two representations. It now follows that $I + J = R$. As $I + J = R$ and $I \cap J = (0)$, clearly $I = (e)$ for some idempotent $e \in R$.

Now let I be any ideal. Then by above observation, we get $\sqrt{I} = (e)$ for some idempotent $e \in R$, and hence $I = (e)$. Thus every ideal is principal and also for each $a \in R$, $(a) = (e)$ for some idempotent $e \in R$. Hence R is Noetherian regular.

Remark 2 : In a principal ideal ring R, every radical ideal need not have a unique representation as an intersection of prime ideals. For example, in Z the zero ideal is a prime ideal which can also be expressed as the intersection of all non-zero prime ideals of Z. This shows that regularity is essential in the above theorem. We also note that semiprime rings need not satisfy the condition (*).

Remark 3 : Any ring R satisfying (*) need not be semiprime. For, in the ring of integers modulo 4, the ideal $(\bar{2})$ is the only radical prime ideal and $\bar{2}$ is a nonzero nilpotent element.

Definition—A nonzero element $a \in R$ is said to be an atom if for each $x \in R$, $x^n a^m = 0$ for some positive integers n, m or $a^n = xy$ for some positive integer n and for some $y \in R$.

Lemma 2—A nonzero element $a \in R$ is an atom if and only if $[a)$ is a maximal saturated set in R.

PROOF : Suppose 'a' is an atom. Let $[a) \subseteq F$ for some proper saturated set F of R. Let $x \in F$. Then $x^n a^m \neq 0$ for all $n, m \in Z^+$ so that $a^n = xy$ for some $n \in Z^+$ and $y \in R$. This shows that $x \in [a)$ and hence $F = [a)$; i.e., $[a)$ is a maximal saturated set.

Conversely assume that $[a)$ is a maximal saturated set. Let $x \in R$. If $x \in [a)$ then $a^n = xy$ for some $y \in R$ and so we are through. If $x \notin [a)$ then $[a) \vee [x) = [0)$ so that $0 = fg$ for some $f \in [a)$ and $g \in [x)$. Now $fy_1 = a^m$ and $gy_2 = x^n$ for some $y_1, y_2 \in R$ so that $x^n a^m = 0$ for some $n, m \in Z^+$. This shows that 'a' is an atom.

Lemma 3—If every prime ideal of R is a principal ideal generated by some idempotent element then every ideal is a principal ideal generated by some idempotent element.

Proof follows by applying Zorn's lemma.

Theorem 3—The following statements are equivalent :

(i) R is a Noetherian regular ring;

(ii) R is semiprime and every maximal saturated set is principally generated by some idempotent element;

(iii) R is semisimple and every maximal saturated set is principal.

PROOF : (i) $\Rightarrow$ (ii). Suppose (i) holds. Clearly R is semiprime. Let F be a maximal saturated set. Then by Theorem 1, $R-F$ is a maximal ideal and so $R-F=(e)$ for some idempotent $e \in R$. We observe that $1-e \in F$. We now show that $1-e$ is an atom. Let $x \in R$. Suppose $x(1-e) \neq 0$. Then $x \not\in (e)$ and so $(x)+(e) = R$ since (e) is a maximal ideal. Now $1 = xy + ey$, for some $y, y_1 \in R$ so that $1-e = x(1-e)y$. This shows that $1-e$ is an atom. It now follows from Lemma 2 that $F = [1-e)$ where $1-e$ is an idempotent.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $a \in \cap \{M \mid M$ is a maximal ideal of $R\}$. Assume $a \neq 0$. Then $[a) \subseteq [e)$ for some maximal saturated set $[e)$ where e is an idempotent. Clearly $1-e$ is not a unit and hence there exists a maximal ideal M such that $1-e \in M$. Now $aa_1 = e$ for some $a_1 \in R$. Since $a \in M$ it follows that $e \in M$ and hence we get $1 \in M$, a contradiction. Therefore we must have $a = 0$ which shows that R is semisimple and hence (iii) follows.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. We first show that R is regular. Lex $x \in R$. Write $T(x) = \{y \in R \mid xy = 0\}$ and $D(x) = \{y \in R \mid (x)+(y) = R\}$. If $T(x) \cap D(x) \neq \phi$ then we are done. Assume $T(x) \cap D(x) = \phi$. By the same argument as before, we can get a maximal saturated set F such that $D(x) \subseteq F$ and $x \in F$. As F is a maximal saturated set, $F = [a)$ for some $a \in R$. Since R is semisimple, $P + (a) = R$ for some proper ideal P of R. Also $P + (a^n) = R$ for any $n \in Z^+$. In fact we can have $(p) + (a^n) = R$ for some $p \in P$. Clearly $p \not\in F$. Since F is a maximal saturated set, R being semiprime we get ge $ap = 0$. Also $x \in F$ so that $(a^n) \subseteq (x)$ and hence $(x) + (p) = R$; i.e., $p \in D(x) \subseteq F$. This shows that $0 = ap \in F$, a contradiction. Therefore $T(x) \cap D(x) \neq \phi$; i.e., R is regular.

Finally we show that every prime ideal is principal. Let P be a prime ideal. Then $R-P$ is a maximal saturated set so that $R-P = [a)$ for some $a \in R$. Since R is regular $(a) = (e)$ for some idempotent $e \in R$. We observe that $P = (1-e)$ and therefore by Lemma 3 every ideal is a principal ideal generated by some idempotent element. Hence R is a Noetherian regular ring.

Remark 4 : In (ii) of the above theorem the condition that "every maximal saturated set is principally generated by an idempotent element" is essential. Also in (iii), the semisimplicity of the ring is essential. These facts are illustrated in the following example

Example—Consider $Z_{(p)}$, the localization of the ring of integers at some prime ideal (p). Clearly the local ring $Z_{(p)}$ is semiprime but not semisimple. It can be easily verified that p as an element in $Z_{(p)}$ is an atom in $Z_{(p)}$. Obviously p is not an idempotent in $Z_{(p)}$. In view of Lemma 2, $[p)$ is a maximal saturated set in $Z_{(p)}$. Since $Z_{(p)}$ is an integral domain, it follows that this is the only maximal saturated set in $Z_{(p)}$.

Definition—An element $x \in R$ is said to be semiprimary if $\sqrt{(x)}$ is a proper prime ideal in R.

Lemma 4—Let R be a semiprime ring in which every non-unit is a zero divisor. Let $x \in R$ be a semiprimary element. If there exists a non-zero element $y \in R$ such that $xy = 0$ then $(x) + (y) = R$. Moreover such a y is an atom in R.

PROOF : Suppose $(x) + (y) \neq R$. Then $x + y$ is a non-unit and so $(x+y)\, d = 0$ for some $d \neq 0$. Now $yd \in (x) \subseteq \sqrt{(x)}$ and so either $y \in \sqrt{(x)}$ or $d \in \sqrt{(x)}$. If $y \in \sqrt{(x)}$ then $y^n = xx_1$ for some $n \in Z^+$ and $x_1 \in R$. Clearly $y^{n+1} = 0$. Since R is semiprime we get $y = 0$, a contradiction. Therefore $y \notin \sqrt{(x)}$ and so $d \in \sqrt{(x)}$. Now $d^m = xx_2$ for some $m \in Z^+$ and $x_2 \in R$ and so $d^m y = yxx_2 = 0$. It follows that $dy = 0$ so that $xd = 0$. Thus we have $0 = xd^m = xxx_2 = x^2 x_2$ from which it follows that $xx_2 = 0$; i.e., $d = 0$, again a contradiction. Therefore $(x) + (y) = R$.

Now we prove that y is an atom. Let $z \in R$ and $zy \neq 0$. As $xzy = 0$ and $zy \neq 0$, by above argument we must have $(x) + (zy) = 0$. Therefore $1 = xx_1 + zyy_1$ so $y = yxx_1 + zy^2 y_1$ which shows that y is an atom.

Theorem 4—R is a Noetherian regular ring if and only if R is semiprime, every non-unit is a zero-divisor and the zero ideal is a product of a finite number of principal ideals generated by semiprimary elements.

PROOF : One implication is evident.

Conversely assume the stated conditions. Then $0 = a_1 a_2 \ldots a_n$ where a_i's $\in R$ and a_i's are semiprimary elements. Since each a_i is a non-unit, there exist $b_i \in R$ such that $a_i b_i = 0$ with $b_i \neq 0$ for $i = 1, 2, \ldots, n$. By the above lemma we have for each i, $(a_i) + (b_i) = R$ and each b_i is an atom. Clearly $(a_i) \cap (b_i) = 0$ and so we get $(a_i) = (e_i)$ and $(b_i) = (f_i)$ for some idempotents e_i, f_i in R. It is easy to see that each e_i is a semiprimary element. Also $e_i f_i = 0$ ($f_i \neq 0$) so that f_i is an atom. Now $\sum_{i=1}^{n} (f_i) + (a_i) = R$ for $i = 1, 2, \ldots, n$. Therefore $\sum_{i=1}^{n} (f_i) + (a_1 \ldots a_n) = \sum_{i=1}^{n} (f_i) + (0)$

$= R$; i.e., $\sum_{i=1}^{n} (f_i) = R$. Without loss of generality we may assume that $f_i \neq f_j$ for $i \neq j$. Since f_i's are distinct idempotent atoms we get $f_i f_j = 0$ for $i \neq j$ so that $\sum_{i=1}^{n} (f_i) = (\sum_{i=1}^{n} f_i) = R$. Consequently $\sum_{i=1}^{n} f_i = 1$. Now let F be a maximal saturated set. Then $\sum_{i=1}^{n} f_i = 1 \in F$ and so by Lemma 1, $f_i \in F$ for some i. Again since f_i is an atom, by Lemma 2, $[f_i)$ is a maximal saturated set and therefore $F = [f_i)$ is a principal saturated set. Thus every maximal saturated set is a principal saturated set generated by an idempotent element. Hence by Theorem 3, R is a Noetherian regular ring. Hence proof of the theorem.

Remark 5 : In the above theorem, the condition that "R is semiprime" is essential. This can be seen by considering Z_4, the ring of integers modulo 4. In Z_4, 0 is a semiprimary element and $\bar{2}$ is the only non-unit which is a zero-divisor. But Z_4 is not regular.

Remark 6 : By considering the ring of integers, we can see that the condition "every non-unit is a zero divisor" cannot be dropped.

Remark 7 : It is well known that $\mathscr{P}(X)$, where X is an infinite set, is a Boolean ring which is not Noetherian. It is semiprime and every non-unit in $\mathscr{P}(X)$ is a zero-divisor. However, the zero ideal cannot be written as a finite product of principal ideals generated by semiprimary elements.

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