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ON A QUATERNION SUBMANIFOLDS OF CO-DIMENSION-2

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In this paper we have studied some properties of a Quaternion submanifolds of co-dimension-2. We have shown that a (1, 1)-tensor field defined by $f=B^{-1} F B$ possess the (f, g, u, v, λ) structure. We have also shown that tensor field defined by $f''' = \frac{F^* + G^* + H^*}{\sqrt{3}}$ admits an almost complex structure.

1. PRELIMINARIES

Integrability conditions of an almost quaternion structure have been obtained by Yano and Kon[1]. Upadhyay[2] has studied some theorems on metric-3 structure. Hamoui[3] has shown that a submanifold of co-dimension-2 admits (F, U, V, u, v, λ) $(G, U', V', u', v', \lambda)$, $(H, U'', V'', u'', v'', \lambda)$ and a most general structure. In this paper we have shown that a (1, 1), tensor field defined by $f = B^{-1} FB$ also possess the (f, g, u, v, λ) structure. Some other important results have also been obtained.

A Quaternion manifold M^{4n} admlts a set of 3-tensor F^*, G^*, H^* of type (1,1), satisfying[1]:

$$F^{*2} = -I, \; G^{*2} = -I, \; H^{*2} = -I \qquad \ldots(1.1)$$

(a) $F^* = G^* H^* = -H^* G^*$

(b) $G^* = F^* H^* = -H^* F^*$.

(c) $H^* = F^* G^* = -G^* F^*$. ...(1.2)

Let g^* be the Hermitian matric then, we have[3]:

$$g^* (F^* X^*, F^* Y^*) = g^* (X^*, Y^*) \qquad \ldots(1.3)$$

where X^*, Y^* are arbitrary vector fields in M^{4n}.

2. STRUCTURES IN M^{4n-2}

Let B represent the immersion say $i : M^{4n-2} \to M^{4n}$.

If C and D are mutually orthogonal unit normals. Then transformation $F^* BX$ of BX by F^* is expressed by[3] :

$$F^* BX = BFX + u(X) C + v(X) D. \quad \ldots(2.1)$$

Here X is an arbitrary vector field of M^{4n-2} & $u.\ v$, are 1 forms. F is a (1, 1) tensor field of M^{4n-2}. Since $F^* C$ & $F^* D$ are orthogonal to C & D respectively, hence

$$\left.\begin{aligned} &\text{(a)} \quad g^*(F^* C, C) = -g^*(C, F^* C) = 0 \\ &\text{(b)} \quad g^*(F^* D, D) = -g^*(D, F^* D) = 0. \end{aligned}\right\} \quad \ldots(2.2)$$

Now corresponding to structures F^*, G^*, H^*, the vector fields U, U', U'', V, V', V'' and form $u, u', u'', v, v'\ v''$ and a function λ such that[3]

$$\left.\begin{aligned} &\text{(a)} \quad F^* C = -BU + \lambda D \\ &\text{(b)} \quad F^* D = -BV - \lambda C. \end{aligned}\right\} \quad \ldots(2.3)$$

$$\text{(a)} \quad G^* BX = BGX + u'(X) C + v'(X) D$$

$$\text{(b)} \quad G^* C = -BU' + \lambda D$$

$$\text{(c)} \quad G^* D = -BV' - \lambda C. \quad \ldots(2.4)$$

$$\text{(a)} \quad H^* BX = BHX + u''(X) C + v''(X) D$$

$$\text{(b)} \quad H^* C = -Bu'' + \lambda D.$$

$$\text{(c)} \quad H^* D = -Bv'' - \lambda C. \quad \ldots(2.5)$$

If E be the Induced Riemannian connection in M^{4n-2} then we have Gauss and Weingarten equations as follows :

$$D^*_{BX} BY = BE_x Y + M(X, Y) C + L(X, Y) D \quad \ldots(2.6)$$

where C & D are symmetric bilinear functions in M^{4n-2}.

$$D^*_{BX} \ D = -B\,'L(X) - 'K(X) C \quad \ldots(2.7)$$

where KX is third fundamental tensor

$$\text{(a)} \quad g('M(X), Y) = M(X, Y); \quad \ldots(2.8)$$

$$\text{(b)} \quad g('L(X), Y) = L(X, Y) \quad \ldots(2.9)$$

since B is the Jacobian map of i

such that B ; $T_p(M^{4n-2}) \to T_{ip}(M^{4n})$.

Suppose a tensor field N on M^{4n} which does not belong to $T(M^{4n})$ so N is now every where tangent to M^{4n-2}. Since B is one-one, B^{-1} also exists and a form N^* in M^{4n} (Sinha and Sharma[8]).

$$BB^{-1} = I,\ B^{-1}B = -I + N^* \otimes N,$$

$$N^* B = 0,\ B^{-1}N = 0,\ N^* N = 1. \qquad ...(2.10)$$

3. Submanifolds of Almost Quaternion Structure

Theorem 3.1—Let M^{4n-2} be a submanifold in a quaternion manifold M^{4n}. M^{4n}. Then a tensor field f admits (f, U, u, v, λ) —Structure.

Proof : Let us put[8]

$$f \text{ def } B^{-1} F^* B,$$

then

$$f^2(X) = B^{-1} F^* BB^{-1} F^* BX: \qquad ...(3.1)$$

which in view of (2.1), (2.20) and (1.1) yields

$$f^2(X) = -X + u(X)U + v(X)V \qquad ...(3.2)$$

also using (2.1), (9.3), (2.10) and (1.1), we can easily show that

$$u(fX) = \lambda v(X),\ v(fX) = -\lambda u(X)$$

$$f(U) = -\lambda V, f(V) = -\lambda U, v(U) = 0$$

$$u(U) = 1 - \lambda^2,\ v(V) = 1 - \lambda^2, u(V) = 0.$$

Theorem 3.2—The submanifolds M^{4n-2} of a quaternion manifold M^{4n} admits the following structure

$$(f', U', V', u', v', \lambda); (f'', U'', V'', u'', v'', \lambda)$$

Proof : Let us define $f' = B^{-1} G^* B, f'' = B^{-1} H^* B$ respectively and using (1.1), (2.1), (2.3), (2.4), (2.5) and (2.10), this can be easily shown.

Theorem 3.3—The tensor field f''' defined by

$$''' = \frac{F^* + G^* + H^*}{\sqrt{3}},\ \text{admits an almost complex structure.}$$

Proof : We can easily show by using (1,2) that $f'''^2 = -I$

$$\begin{aligned} g^*(f'''X, f'''Y) = 1/3\,[&g^*(F^*X, F^*Y) + g^*(G^*X, G^*Y) \\ &+ g^*(H^*X, H^*Y) + g^*(F^*X, G^*Y) \\ &+ g^*(F^*X, H^*Y) + g^*(GX, F^*Y) \\ &+ g^*(G^*X, H^*Y) + g^*(H^*X, F^*Y) \\ &+ g^*(H^*X, G^*Y)]. \end{aligned}$$

Making use of (1.1) – (1.3) & (2.1); we get

$$g^*(f'''X, f'''Y) = g^*(X, Y) \text{ and } g^*(fX, fY) = g^*(FX, FY).$$

4. General Proerties

Theorem 4.1—In M^{n-2} submanifold of a quaternion manifold

M^{4n}, If $(E_X\ F)(Y) = u(X)\ 'MY - v(Y)'\ L(X)$, then

$(D^*_{BX}\ F^*)(BY) - (D^*_{BY})(BX) = 0$, and similar results for G^* and H^*

Proof :—Using (2.1), (2,2) − (2.8), (2.10) and making use of $(E_X\ F)(Y) = v(Y)\ 'L(X) - u(X)\ 'M(Y)$, we find

$$(D^*_{BX}\ F^*)(BY) - (D^*_{BY}\ F^*)(BX) = 0. \quad \ldots(4.1)$$

Similarly it can be shown for G^* & H^* structures.

Theorem 4.2—In a submanifold M^{4n-2} of a quaternion manifold M^{4n}, we have

$$N^*\ D^*_{BX}\ F^*\ C = 0,\ N^*\ D^*_{BX}\ G^*\ C = 0,\ N^*\ D^*_{BX}\ H^*\ C = 0;$$

and

$$N^*\ D^*_{BX}\ F^*\ D = 0,\ N^*\ D^*_{BX}\ G^*\ D = 0,\ N^*\ D^*_{BX}\ H^*\ C = 0.$$

Proof : Using (2.3) — (2.2), (2.7) & (2.8) we get

(a) $B^{-1}(D^*_{BX}\ F^*\ C) = -E_X\ U - \lambda' L(X)$

(b) $B^{-1}\ (D^*_{BX}\ F^*\ D = -E_X\ V - \lambda' M(X)$

(c) $B^{-1}\ (D^*_{BX}\ G^*\ C) = -E_X\ U' - \lambda' L(x)$

(d) $B^{-1}\ (D^*_{BX} G^*\ D) = -E_x\ V' - \lambda' M(X)$

(e) $B^{-1}\ (D^*_{BX}\ H^*\ C) = -E_X\ U'' - \lambda' L(X)$

(f) $B^{-1}(D^*_{BX}\ H^*\ D) = -E_X\ V'' - \lambda' L(X)$ ...(4.2)

using these equation and keeping in view of (2.10) the theorem follows.

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