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ON ALMOST CONTINUOUS FUNCTIONS

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Some characterizations of almost continuous functions in the sense of Singal[16] are obtained. It is shown that every nearly almost open and almost weakly continuous function is almost continuous in the sense of Husain[6].

1. INTRODUCTION

Singal and Singal[16] introduced and investigated the notion of almost continuous functions. Husain[6] introduced the notion of almost continuous functions. Long and Carnahan[10] pointed out that these two notions of almost continuous functions are independent of each other. The purpose of the present paper is to obtain some characterizations of almost continuity in the sense of Singal and to show that every nearly almost open and almost weakly continuous function is almost continuous in the sense of Husain.

2. PRELIMINARIES

Throughout the present paper, X and Y always mean topological spaces and by $f: X \to Y$ we denote a single valued function. Let A be a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be α-open[12] (resp. preopen[11]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$). A subset A is said to be semi-open[9] (resp. semi-preopen[3]) if there exists an open (resp. preopen) set U such that $U \subset A \subset \mathrm{Cl}(U)$. It is shown that a subset A is semi-open (resp. semi-preopen) if and only if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The family of all α-open (resp. semi-open, preopen and semi-preopen) sets of X is denoted by $\alpha(X)$ (resp. $SO(X)$, $PO(X)$ and $SPO(X)$).

Lemma 2.1—For a topological space X, the following properties hold :

$$\text{(i) } \alpha(X) = PO(X) \cap SO(X) \text{ and (ii) } PO(X) \cup SO(X) \subset SPO(X).$$

PROOF : This follows easily from the definitions.

The complement of an α-open (resp. semi-open, preopen) set is said to be α-closed (resp. semi-closed, preclosed). The intersection of all α-closed (resp. semi-closed, preclosed) sets containing A is called the α-closure[2] (resp. semi-closure[4], preclosure[5]) and is denoted by α Cl (A) (resp. s Cl (A), P cl (A)). A subset A is said to be regular open if A = Int (Cl (A)). The complement of a regular open set is said to be regular closed.

3. Almost Continuity in the Sense Singal

Definition 3.1—A function $f: X \rightarrow Y$ is said to be almost continuous[16] (briefly a.c.S.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset$ Int (Cl (V)).

Singal and Singal[16] showed that a function $f: X \rightarrow Y$ is a.c.S. if and only if $f^{-1}(V)$ is open (resp. closed) in X for every regular open (resp. regular closed) set V of Y. This characterization is very useful and will be utilized in the sequel.

Theorem 3.2—The following are equivalent for a function $f: X \rightarrow Y$:

(a) f is a.c.S.

(b) Cl $(f^{-1}(V)) \subset f^{-1}$ (Cl (V)) for every $V \in SPO(Y)$.

(c) Cl $(f^{-1}(V)) \subset f^{-1}$ (Cl (V)) for every $V \in SO(Y)$.

(d) $f^{-1}(V) \subset$ Int $(f^{-1}$ (Int (Cl (V)))) for every $V \in PO(Y)$.

Proof : (a) ⇒ (b): Let $V \in SPO(Y)$. By Theorem 2.4 of Andrijević[3], Cl (V) is regular closed in Y. Since f is a.c.S., f^{-1} (Cl (V)) is closed in X and we obtain Cl $(f^{-1}(V)) \subset f^{-1}$ (Cl (V)).

(b) ⇒ (c) : Since $SO(Y) \subset SPO(Y)$, this is obvious.

(c) ⇒ (a) : Let F be any regular closed set of Y. Then F = Cl (Int (F)) and hence $F \in SO(Y)$. Therefore, we have Cl $(f^{-1}(F)) \subset f^{-1}$ (Cl (F)) $= f^{-1}(F)$. Hence $f^{-1}(F)$ is closed and f is a.c.S.

(a) ⇒ (d) : Let $V \in PO(Y)$. Then $V \subset$ Int (Cl (V))) and Int (Cl (V)) is regular open. Since f is a.c.S., f^{-1} (Int (Cl (V))) is open in X and hence $f^{-1}(V) \subset f^{-1}$ (Int (Cl (V))) = Int $(f^{-1}$ (Int (Cl (V)))).

(d) ⇒ (a) : Let V be any regular open set of Y. Then $V \in PO(Y)$ and hence $f^{-1}(V) \subset$ Int $(f^{-1}$ (Int (Cl (V)))) = Int $(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is open in X and hence f is a.c.S.

Lemma 3.3—For a subset V of Y, the following properties hold :

(a) αCl (V) = Cl (V) for every $V \in SPO(Y)$.

(b) P cl (V) = Cl (V) for every $V \in SO(Y)$.

(d) s cl (V) = Int (Cl (V)) for every $V \in PO(Y)$.

Proof : (a) Let $V \in SPO(Y)$. Then $V \subset$ Cl (Int (Cl (V))) and by Theoren 2.2 of Andrijević[2] we have α Cl $(V) = V \cup$ Cl (Int (Cl (V))) = Cl (V).

(b) This follows from Theorem 2.4 of El-Deeb *et al.*[5].

(c) Let $V \in PO(Y)$. Then $V \subset \text{Int}(\text{Cl}(V))$ and by Theorem 1.5 of Andrijević[3], we have $s\,\text{Cl}(V) = V \cup \text{Int}(\text{Cl}(V)) = \text{Int}(\text{Cl}(V))$.

Corollary 3.4—The following are equivalent for a function $f : X \to Y$:

(a) f is a.c.S.

(b) $\text{Cl}(f^{-1}(V)) \subset f^{-1}(\alpha\,\text{Cl}(V))$ for every $V \in SPO(Y)$.

(c) $\text{Cl}(f^{-1}(V)) \subset f^{-1}(P\,\text{cl}(V))$ for every $V \in SO(Y)$.

(d) $f^{-1}(V) \subset \text{Int}(f^{-1}(s\,\text{Cl}(V)))$ for every $V \in PO(Y)$.

PROOF : This is an immediate consequence of Theorem 3.2 and Lemma 3.3.

Long and Carnahan[10] showed that if $f : X \to Y$ is open a.c.S. then $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every open set V of Y. Recently, Allam *et al.*[1] have improved this result as follows if : $f : X \to Y$ is open a.c.S. then $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in PO(Y)$. The following corollary is the further improvement of the previous result.

Corollary 3.5—If a function $f : X \to Y$ is open and a.c.S., then $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in SPO(Y)$.

PROOF: Since f is open, $\text{Cl}(f^{-1}(S)) \supset f^{-1}(\text{Cl}(S))$ for every subset S of Y. Therefore, this follows immediately from Theorem 3.2.

Definition 3.6—A function $f : X \to Y$ is said to be almost open[15] if $f(U) \subset \text{Int}(\text{Cl}(f(U)))$ for every open set U of X.

Mashhour *et al.*[11] called an almost open function preopen. Theorem 11 of Rose[15] states that $f : X \to Y$ is almost open if and only if $f^{-1}(\text{Cl}(V)) \subset \text{Cl}(f^{-1}(V))$ for every open set V of Y. It is shown in Theorem 14 of Rose[15] that $f : X \to Y$ is almost open and a.c.S. if and only if $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every open set V of Y.

Theorem 3.7—A function $f : X \to Y$ is almost open and a.c.S. if and only if $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in SO(Y)$.

PROOF : *Necessity*—Let $V \in SO(Y)$. Since f is a.c.S., by Theorem 3.2 $\text{Cl}(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$. Since f is almost open, we have

$$f^{-1}(\text{Cl}(V)) = f^{-1}(\text{Cl}(\text{Int}(V))) \subset \text{Cl}(f^{-1}(\text{Int}(V))) \subset \text{Cl}(f^{-1}(V)).$$

Therefore, we obtain $\text{Cl}(f^{-1}(V)) = f^{-1}(\text{Cl}(V))$ for every $V \in SO(Y)$.

Sufficiency—It follows from Theorem 11 of Rose[15] that f is almost open. Let F be any regular closed set of Y. Then $F = \text{Cl}(\text{Int}(F))$ and hence $F \in SO(Y)$. By the hypothesis, $\mathrm{Cl}(f^{-1}(F)) = f^{-1}(\mathrm{Cl}(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is closed in X. Therefore, f is a.c.S.

Corollary 3.8—A function $f: X \to Y$ is almost open and a.c.S if and only if $\mathrm{Cl}(f^{-1}(V)) = f^{-1}(\mathrm{Cl}(V))$ for every $V \in \alpha(Y)$.

PROOF : Since $\alpha(Y) \subset SO(Y)$ and every open set is α-open, this follows from Theorem 3.7 and Theorem 14 of Rose[15].

The following question will be raised naturally : can $SO(Y)$ in Theorem 3.7 be replaced by $PO(Y)$? Actually, it is shown in Corollary 2.4 of Allam *et al.*[1] that if $f: X \to Y$ is OPEN a.c.S. then $\mathrm{Cl}(f^{-1}(V)) = f^{-1}(\mathrm{Cl}(V))$ for every $V \in PO(Y)$. However, the answer is negative under the condition that f is almost open a.c.S. as the following example shows.

Example 3.9—Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a, b, c\}, \{a, c, d\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a\}, \{c\}\}$. Let $Y = \{x, y, z\}$ and $\sigma = \{Y, \phi, \{x, y\}, \{z\}\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined as follows : $f(a) = x, f(b) = y$ and $f(c) = f(d) = z$. Then f is continuous (hence a.c.S.) almost open but not open. There exists $V = \{y, z\} \in PO(Y)$ such that $\mathrm{Cl}(f^{-1}(V)) \neq f^{-1}(\mathrm{Cl}(V))$.

4. Almost Continuity in the Sense of Husain

Definition 4.1—A function $f: X \to Y$ is said to be almost continuous[6] (briefly a.c.H.) if for each $x \in X$ and each open set V of Y containing $f(x)$, $\mathrm{Cl}(f^{-1}(V))$ is a neighbourhood of x.

It is easily proven that $f: X \to Y$ is a.c.H. if and only if $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$, that is, $f^{-1}(V) \in PO(X)$ for every open set V of Y. Mashhour *et al.*[11] called a.c.H. functions *precontinuous.*

Definition 4.2—A function $f: X \to Y$ is said to be almost weakly continuous[7] if $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$ for every open set V of Y.

Levine[8] defined $f: X \to Y$ to be weakly continuous if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \mathrm{Cl}(V)$ and showed that $f: X \to Y$ is weakly continuous if and only if $f^{-1}(V) \subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$ for every open set V of Y. Therefore, almost weak continuity is implied by both weak continuity and almost continuity in the sense of Husain.

Theorem 4.3—The following are equivalent for a function $f: X \to Y$:

(a) f is almost weakly continuous.

(b) $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(\mathrm{Cl}(V))$ for every $V \in PO(Y)$.

(c) $P\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every $V \in PO(Y)$.

(d) $P\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y.

PROOF : (a) $\Rightarrow$ (b) : Let $V \in PO(Y)$. By utilizing Theorem 3.1 of Noiri[13], we obtain $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset \mathrm{Cl}(\mathrm{Int}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V))))) \subset f^{-1}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))) = f^{-1}(\mathrm{Cl}(V))$.

(b) $\Rightarrow$ (c) : Let $V \in PO(Y)$. By utilizing Theorem 1.5 of Andrijević[3], we obtain $P\mathrm{cl}(f^{-1}(V)) = f^{-1}(V) \cup \mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset f^{-1}(\mathrm{Cl}(V))$.

(c) $\Rightarrow$ (d) : This is obvious since every open set is preopen.

(d) $\Rightarrow$ (a) : Let V be any open set of Y. We have $\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))) \subset P\,\mathrm{cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ and hence by Theorem 3.1 of Noiri[13] f is almost weakly continuous.

Definition 4.4—A function $f : X \to Y$ is said to be nearly almost open[14] if there exists an open basis $\mathcal{B}$ for the topology on Y such that $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$ for every $V \in \mathcal{B}$.

Every almost open function is nearly almost open but not conversely by Example 3 of Rose[14]. Rose[15] showed that every almost open weakly continuous function is a.c.H. Moreover, Rose[14] showed that nearly almost open weakly continuous functions are a.c.H. On the other hand, Noiri[13] showed that every almost open almost weakly continuous function is a.c.H. The following theorem is an improvement of the previous results.

Theorem 4.5—If a function $f : X \to Y$ is nearly almost open and almost weakly continuous, then f is a.c.H.

PROOF : Since f is nearly almost open, there exists an open basis $\mathcal{B}$ for the topology on Y such that $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$ for every $V \in \mathcal{B}$. Let W be any open set of Y. There exists a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $W = \cup\{V \mid V \in \mathcal{B}_0\}$. Therefore, we obtain

$$f^{-1}(W) = \bigcup_{V\in\mathcal{B}_0} f^{-1}(V) \subset \bigcup_{V\in\mathcal{B}_0} \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V)))) \subset \bigcup_{V\in\mathcal{B}_0}$$

$$\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))) \subset \mathrm{Int}(\mathrm{Cl}(\bigcup_{V\in\mathcal{B}_0} f^{-1}(V)))$$

$$= \mathrm{Int}(\mathrm{Cl}(f^{-1}(W))).$$

This shows that $f^{-1}(W) \in PO(X)$ and hence f is a.c.H.

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