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ON THE MULTIVALENT FUNCTIONS

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The object of the present paper is to derive some sufficient conditions for p-valently close-to-convexity, p-valently starlikeness and p-valently convexity.

1. INTRODUCTION

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N)$$

which are regular in $D = \{z \mid |z| < 1\}$.

A function $f(z)$ in $A(p)$ is said to be p-valently convex if and only if

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} > 0 \text{ in } D.$$

We denote by $C(p)$ the subclass of $A(p)$ consisting of all p-valently convex functions in D.

A function $f(z)$ in $A(p)$ is said to be p-valently starlike if and only if

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \text{ in } D.$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of all functions which are p-valently starlike in D.

A function $f(z)$ in $A(p)$ is said to be p-valently close-to-convex, if there exists a p-valently starlike function $g(z) \in A(p)$ for which $f(z)$ satisfies the following

$$\operatorname{Re} \frac{z f'(z)}{g(z)} > 0 \text{ in } D.$$

We denote by $K(p)$ the subclass of $A(p)$ consisting of all functions which are p-valently close-to-convex in D.

Every p-valently starlike function is p-valently close-to convex, and every p-valently close-to-convex function is p-valent in D[2,10,4]. Ozaki[7] (Theorem 3) proved that if

$f(z) \in A(p)$ and

$$1 + \mathrm{Re}\, \frac{z f''(z)}{f'(z)} < \frac{k + p + 1}{2} \quad \text{in } D$$

then $f(z)$ is at most k-valent in D.

Moreover, by using Umezawa's[11] result (Theorem 6), we have that if $f(z) \in A(p)$ and

$$1 + \mathrm{Re}\, \frac{z f''(z)}{f'(z)} < p + \tfrac{1}{2} \text{ in } D$$

then $f(z)$ is convex of order at most p in one direction in D, and at most p-valent in D.

2. Preliminaries

Lemma 1—Let $f(z) \in A(p)$ and if there exists a $(p - k + 1)$-valently starlike function $g(z) = \sum_{n=n-p+1}^{\infty} b_n z^n$, $(b_{p-k+1} \neq 0)$ that satisfies

$$\mathrm{Re}\, \frac{z f^{(k)}(z)}{g(z)} > 0 \text{ in } D$$

then $f(z)$ is p-valently close-to-convex in D.

Proof : From Theorem 8 of Nunokawa[4], we have

$$\mathrm{Re}\, \frac{z f'(z)}{G(z)} > 0 \text{ in } D$$

where $G(z)$ is p-valently starlike in D.

This shows that $f(z) \in K(p)$ and $f(z)$ is p-valently close-to-convex in D.

Lemma 2—Let $f(z) \in A(p)$ and suppose that there exists a positive integer k for which

$$\left| \arg \frac{f^{(k)}(z)}{z^{p-k}} \right| < \frac{\pi}{2} \alpha \text{ in } D \qquad \ldots(1)$$

where $1 \leq k \leqq p$ and $0 < \alpha \leq 1$.

Then we have

$$\left| \arg \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} \right| < \pi \alpha \text{ in } D.$$

Proof : We easily have

$$\frac{f^{(k-1)}(z)}{z f^{(k)}(z)} = \int_0^1 \frac{f^{(k)}(tz)}{f^{(k)}(z)}\, dt$$

(*equation continued on p. 579*)

$$= \frac{z^{p-k}}{f^{(k)}(z)} \int_0^1 t^{p-k} \frac{f^{(k)}(tz)}{(tz)^{p-k}} \, dt \qquad \ldots(2)$$

and it easily follows that

$$\left| \arg t^{p-k} \frac{f^{(k)}\, tz}{(tz)^{p-k}} \right| = \left| \arg \frac{f^{(k)}\, tz}{(tz)^{p-k}} \right| < \frac{\pi}{2} \alpha \text{ in } D$$

for $0 \leqq t \leqq 1$.

Then the integral

$$\int_0^1 t^{p-k} \frac{f^{(k)}(tz)}{(t\,z)^{p-k}} \, dt$$

lies in the same convex sector $\{w \| \arg w | < \frac{\pi}{2} \alpha\}$.

Therefore, from (1) and (2), we have

$$\left| \arg \frac{f^{(k-1)}(z)}{zf^{(k)}(z)} \right| \leqq \left| \arg \frac{z^{p-k}}{f^{(k)}(z)} \right| + \left| \arg \int_0^1 t^{p-k} \frac{f^{(k)}\, tz}{(tz)^{p-k}} \, dt \right|$$

$$< \frac{\pi}{2}\alpha + \frac{\pi}{2}\alpha = \pi \alpha \text{ in } D.$$

This completes our proof.

Lemma 3—Let $f(z) \in A(p)$ and suppose there exists a positive integer k for which

$$k + \mathrm{Re} \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} > 0 \text{ in } D$$

where $1 \leqq k \leqq p$.

Then we have

$$k - 1 + \mathrm{Re} \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} > 0 \text{ in } D.$$

We owe this lemma to Nunokawa[4] (Lemma 9).

3. Statement of Results

Theorem 1—Let $f(z) \in A(p)$ and suppose that the exists a positive integer k for which

$$k + \mathrm{Re}\ \frac{z f^{(k-1)}(z)}{f^{(h)}(z)} < \beta \text{ in } D$$

where $1 \leqq k \leqq p$ and $p < \beta \leqq p + \frac{1}{2}$.

Then we have $f(z) \in C(p)$ or $f(z)$ is p-valently close-to-convex in D.

PROOF : Let us put

$$H(z) = \frac{1}{\beta - p}\left\{\beta - k - \frac{z f^{(k\ 1)}(z)}{f^{(k)}(z)}\right\} = \frac{z g'(z)}{g(z)}.$$

Then we have $H(0) = 1$ and $\mathrm{Re}\ H(z) > 0$ in D. This shows that $g(z)$ is univalently starlike in D.

Applying the same method as in Nunokawa and Owa[5,6], we easily have

$$\frac{1}{\beta - p} \log \frac{z^{p-k}}{f^{(k)}(z)}\ {}_pA_k = \log \frac{g(z)}{z}$$

where ${}_pA_k = p(p-1)(p-2)\ldots(p-k+1)$.

It follows that

$$\frac{f^{(k)}(z)}{{}_pA_k\, z^{p-k}} = \left(\frac{g(z)}{z}\right)^{p-\beta}.$$

Applying the result by Komatu[7] and Robinson[8], we have

$$\frac{f^{(k)}(z)}{{}_pA_k\, z^{p-k}} = \left(\frac{g(z)}{z}\right)^{p-\beta} \prec \left(\frac{1}{1-z}\right)^{2(p-\beta)} \text{ in } D$$

where the symbol $\prec$ denotes subordination.

Then we have

$$\left|\arg \frac{f^{(k)}(z)}{{}_pA_k\, z^{p\ k}}\right| = \left|\arg \frac{f^{(k)}(z)}{z^{p-k}}\right| \leqq 2(\beta - p)\sin^{-1}|z|$$

$$< 2(\beta - p)\frac{\pi}{2} \leqq \frac{\pi}{2} \text{ in } D.$$

This shows that

$$\mathrm{Re}\frac{f^{(k)}(z)}{z^{p-k}} = \mathrm{Re}\ \frac{z f^{(k)}(z)}{z^{p-k+1}} > 0 \text{ in } D. \qquad \ldots(3)$$

On the other hand, $g(z) = z^{p-k+1}$ is $(p-k+1)$-valently starlike in D and therefore we have

$$\mathrm{Re}\ \frac{z f^{(k)}(z)}{g(z)} > 0 \text{ in } D.$$

From Lemma 1, we have $f(z) \in K(p)$. This completes our proof.

Remark 1 : Applying the same method as in the proof of Theorem 6 of Nunokawa[3] and from (3), we can also prove Theorem 1.

Theorem 2—Let $f(z) \in A(p)$ and suppose that there exists a positive integer k for which

$$k + \operatorname{Re} \frac{z f^{(k+1)}(z)}{f^{(k)}(z)} < \beta \text{ in } D$$

where $2 \leqq k \leqq p$ and $p < \beta \leqq p + \frac{1}{4}$.

Then we have $f(z) \in C(p)$ and $f(z) \in S(p)$, or $f(z)$ is p-valently convex in D and $f(z)$ is p-valently starlike in D.

PROOF : Applying the same method as in the proof of Theorem 1, we have

$$\frac{f^{(k)}(z)}{pA_k z^{p-k}} = \left(\frac{g(z)}{z} \right)^{p-\beta} \prec \left(\frac{1}{1-z} \right)^{2(p-\beta)} \text{ in } D. \qquad ...(4)$$

From the assumption of Theorem 2 and (4), we have

$$\left| \arg \frac{f^{(k)}(z)}{pA_k z^{p-k}} \right| = \left| \arg \frac{f^{(k)}(z)}{z^{p-k}} \right|$$

$$< \frac{\pi}{2} 2(\beta - p) \leqq \frac{\pi}{4} \text{ in } D. \qquad ...(5)$$

From Lemma 2 and (5), we have

$$\left| \arg \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} \right| < \frac{\pi}{2} \text{ in } D.$$

This shows that

$$\operatorname{Re} \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} > 0 \text{ in } D$$

and it follows that

$$k - 1 + \operatorname{Re} \frac{z f^{(k)}(z)}{f^{(k-1)}(z)} > 0 \text{ in } D.$$

Applying Lemma 3 over again, we have

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} > 0 \text{ in } D$$

and

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \text{ in } D.$$

This comletes our proof.

Applying the same method as in the proof of Theorem 2, we have

Theorem 3—Let $f(z) \in A(p)$ and suppose

$$1 + \text{Re}\,\frac{zf''(z)}{f'(z)} < p + \tfrac{1}{4} \text{ in } D$$

where $2 \leqq p$.

Then $f(z)$ is p-valently starlike in D.

Remark 2 : Singh and Singh[9] (Theorem 6) proved the following theorem :

Theorem — If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in D and

$$1 + \text{Re}\,\frac{zf''(z)}{f'(z)} < \tfrac{3}{2} \text{ in } D$$

then $f(z)$ is starlike in D.

The proof of this theorem is simple but it is not generalized to multivalent functions. Therefore, the author gives finally the following conjecture :

Conjecture—Let $(z) \in A(p)$ and suppose

$$1 + \text{Re}\,\frac{zf''(z)}{f'(z)} < p + \tfrac{1}{2} \text{ in } D.$$

Then $f(z)$ is p-valently starlike in D.

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