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THE HANKEL-CLIFFORD TRANSFORMATION ON CERTAIN SPACES OF ULTRADISTRIBUTIONS

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In this paper we introduce certain spaces of testing functions ${}^{p}S^{\beta}_{\mu,\alpha}$ contained in H_μ. The elements of the dual spaces are ultradistributions. The Hankel-Clifford transform h_μ for $\mu \geqslant 0$ is a continuous linear operator in spaces of these type. The generalized Hankel-Clifford transform h'_μ is defined as a continuous linear mapping between the dual spaces. The developed theory is applied to find classes of existence of classical and generalized solutions for a Cauchy problem of the Bessel type operator $B_\mu = Dx^{\mu+1} Dx^{-\mu}$.

1. INTRODUCTION

Méndez[5] introduced the space of testing function H_μ that consists of all infinitely differentiable functions ψ on $I = (0, \infty)$ such that

$$\sup_{x \in I} | x^m D^n (x^{-\mu} \psi (x)) | < \infty \qquad \text{for every } m, n \in N$$

and its dual H'_μ to extend the classical Hankel-Clifford transformation to distributions of slow growth. More recently Gonzalez[2] considers some new function spaces, which are similar to the space studied by Lee[4]. The Hankel-Clifford transform acts on these spaces as a continuous linear mapping. Also, Gonzalez[2] analyzes a Cauchy problem for the systems of equations containing the Bessel-type operator $B_\mu = Dx^{\mu+1} Dx^{-\mu}$. He obtains classes of uniqueness and existence of generalized solution for such problem. The definition of a new convolution compatible with the Hankel-Clifford transformation allows to him to obtain an explicit expression of the solutions.

In this paper, according to the ideas of Romieu[7], Gelfand and Shilov[1], Pathak and Pandey[6] and others, we introduce spaces of ultradifferentiable functions, denoted by ${}^{p}S^{\beta}_{\mu,\alpha}$, with a structure similar to the spaces defined by Sanchez[8]. The Hankel-Clifford transform is a continuous linear mapping on this spaces. Therefore, the generalized transformation is also a linear and continuous mapping on the corresponding dual spaces.

We prove some properties of the spaces ${}^{p}S^{\beta}_{\mu,\alpha}$ and study how the differentiable operators D, $P_\mu = x^{\mu+1} D x^{-\mu}$ and B_μ act on them. Moreover, certain spaces of multipliers are defined.

Finally, we consider the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = P(B_\mu)\, u(x,t)$$

$$u(x, t_0) = f(x)$$

where $u(x,t)$ is an unknown vector function, $u(x,t) = \{u_j(x,t)\}$ and P a matrix of polynomials. New classes of existence of classical and generalized functions are obtained.

2. Some Spaces of Testing Functions and Their Duals

In this section we introduce certain spaces of testing functions, subspaces of H_μ, which have a structure similar to those defined by Sànchez[8].

2.1. The Space ${}^{p}S_{\mu,\alpha,A}$

Let $\alpha \geqslant 0$, $\mu \in R$ and $p \in N$. We define the function space ${}^{p}S_{\mu,\alpha,A}$ as the collection of all complex valued smooth functions ψ defined on I such that

$$| x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_{q,\delta} (A + \delta)^m (pm)!^{\alpha}$$

for every $q \in N$ and $\delta > 0$. $C_{q,\delta}$ are constants depending on ψ.

${}^{p}S_{\mu,\alpha,A}$ is a linear space with the usual operations. Moreover, if

$$\|\psi\|_{q,\delta} = \sup_{\substack{x \in I \\ m \in N}} \frac{| x^m D^q (x^{-\mu} \psi(x)) |}{(A+\delta)^m (pm)!^{\alpha}}$$

for every $q \in N$ and $\delta > 0$, each $\| \;\|_{q,\delta}$ is a seminorm on ${}^{p}S_{\mu,\alpha,A}$, and the collection $\Gamma = \{\| \;\|_{q,\delta}\}_{q \in N,\, \delta > 0}$ is a multinorm because each $\| \;\|_{0,\delta}$ is a norm. Since the systems of seminorms Γ and $\Gamma_1 = \{\| \;\|_{q,1/n}\}_{q,\, n \in N}$ are equivalent, the space ${}^{p}S_{\mu,\alpha,A}$ equiped with the topology generated by Γ_1, is a countable multinormed space.

We now list some interesting properties of the space ${}^{p}S_{\mu,\alpha,A}$.

Property 2.1.1—${}^{p}S_{\mu,\alpha,A} \subset H_\mu$, the inclusion being continuous.

Property 2.1.2—${}^{p}S_{\mu,\alpha,A}$ is complete and therefore a Fréchet space.

To prove the last assertion it is enough to use Property 2.1.1, since H_μ is a complete space.

As a consequence of the above results, ${}^{p}S_{\mu,\alpha,A}$ is clearly a space of testing functions. Its dual, $({}^{p}S_{\mu,\alpha,A})'$, is a space of generalized functions.

Property 2.1.3—If $\alpha > 0$, then $D(I) \subset {}^pS_{\mu,\alpha,A}$ and the topology of $D(I)$ is stronger that the topology induced by ${}^pS_{\mu,\alpha,A}$ in $D(I)$.

PROOF : If $\psi \in D(I)$ one has

$$| x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_q (A + \delta)^m (pm)!^\alpha \left(\frac{L}{A+\delta}\right)^m (pm)!^{-\alpha}$$

for $m, q \in N$ and $\alpha > 0$, where $L = \sup \{x : x \in \text{supp}\, \psi\}$ and $C_q = \sup_{0<x<L} |D^q x^{-\mu} \psi(x)|$. Hence

$$| x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_\delta C_q (A + \delta)^m (pm)!^\alpha$$

with $C_\delta > \left(\frac{L}{A+\delta}\right)^m (pm)!^{-\alpha}$, for $m \in N$. Consequently, $D(I) \subset {}^pS_{\mu,\alpha,A}$ (both algebraically and topologically).

From the above result, the nontriviality of ${}^pS_{\mu,\alpha,A}$ follows provided that $\alpha > 0$. This space is dense in $E(I)$.

On the other hand, if $\alpha = 0$ and $\sup_{x \in I} | x^m D^q (x^{-\mu} \psi(x)) | < C_{q,\delta} (A + \delta)^m$, for $\delta > 0$ and $m, q \in N$, then $\psi \in {}^pS_{\mu,0,A}$. Hence, ${}^pS_{\mu,0,A}$ coincides with the space $H_{\mu,0,A}$ defined by Gonzalez[2]. He denoted this space by $B_{\mu,A}$ due to its relation with the space $\beta_{\mu,A}$ introduced by Zemanian[13].

Property 2.1.4—(a) ${}^pS_{\mu,\alpha,A} \subset H_{\mu,\alpha p,p\, p\alpha A}$

(b) If $p > 1$, $H_{\mu,\alpha,A} \subset {}^pS_{\mu,\alpha,A}$

(c) $H_{\mu,\alpha,A} \subset {}^1S_{\mu,r\alpha,A}$ with $r > 1$.

All inclusions are continuous.

Recall that the space $H_{\mu,\alpha,A}$ (Gonzalez[2]) consists of all smooth functions $\psi(x)$ defined on $0 < x < \infty$ such that

$$| x^m D^q (x^{-\mu} \psi(x)) | \leqslant (A + \delta)^m m^{m\alpha} C_{q,\delta}$$

for $\delta > 0$ and $m, q \in N$. Then the proof of the Property 2.1.4 follows simply by using Stirling's formula.

Every inclusion transforms bounded sets into bounded sets, therefore it is continuous.

Property 2.1.5—${}^pS_{\mu+k,\alpha,A}$ is contained in ${}^pS_{\mu,\alpha,A}$, for each $k \in N$, the inclusion being continuous.

PROOF : Assume $k = 1$ and choose $\psi \in {}^pS_{\mu+1,\alpha,A}$. One then has $\sup_{x \in I}$ $| x^m D^q (x^{-\mu} \psi(x)) | \leqslant \sup_{x \in I} | x^{m+1} D^q x^{-\mu-1} \psi(x) | + q \sup_{x \in I} | x^m D^{q-1} x^{-\mu-1} \psi(x) |$

$\leqslant C_{q,s}(A+\delta)^{m+1}((m+1)p)!^{\alpha} + qC_{q-1,s}(A+\delta)^{m}(pm)!^{\alpha}$

for $\delta > 0$, $m \in N$ and $q \in N - \{0\}$. Moreover :

$$\sup_{x\in I} | x^m x^{-\mu} \psi(x) | = \sup_{x\in I} | x^{m+1} x^{-\mu-1} \psi(x) | < C_{0,s}(A+\delta)^{m+1} (p(m+1))!^{\alpha}.$$

Making use of Stirling's formula it can easily be seen that ψ is in ${}^{p}S_{\mu,\alpha,A}$.

The proof is completed by induction on k.

The following result permit one to define a countable union space.

Property 2.1.6—If $0 < A_1 < A_2$ then ${}^{p}S_{\mu,\alpha,A_1} \subset {}^{p}S_{\mu,\alpha,A_2}$, the inclusion being continuous.

Hence, the union space can be defined as

$${}^{p}S_{\mu,\alpha} = \bigcup_{A=1}^{\infty} {}^{p}S_{\mu,\alpha,A}$$

which is endowed with the inductive limit topology.

${}^{p}S_{\mu,\alpha}$ is a space of testing functions and its dual, $({}^{p}S_{\mu,\alpha})'$, is a space of generalized functions.

2.2. *The space* ${}^{p}S_{\mu}^{\beta,B}$.

Let μ be a real number, $\beta \geqslant 0$ and $B > 0$. We define ${}^{p}S_{\mu}^{\beta,B}$ as the space of complex valued smooth functions $\psi(x)$ on $I = (0, \infty)$ such that

$$\sup_{x\in I} | x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_{m,\rho}(B+\rho)^q (pq)!^{\beta}$$

for every $m, q \in N$ and $\rho > 0$. $C_{m,\rho}$ are constants depending on ψ.

It can be easily seen that the set $\Gamma = \{\| \|^{m,\rho}\}_{m\in N,\, \rho>0}$ represents a system of norms on ${}^{p}S_{\mu}^{\beta,B}$. Here

$$\|\psi\|^{m,\rho} = \sup_{\substack{x\in I\\ q\in N}} \frac{| x^m D^q (x^{-\mu} \psi(x)) |}{(B+\rho)^q (pq)!^{\beta}} \quad \text{for every } m \in N \text{ and } \rho > 0.$$

We now study several properties of the space ${}^{p}S_{\mu}^{\beta,B}$.

Property 2.2.1—${}^{p}S_{\mu}^{\beta,B}$ is contained in H_μ and the inclusion is continuous.

Property 2.2.2—${}^{p}S_{\mu}^{\beta,B}$ is a Fréchet space.

${}^{p}S_{\mu}^{\beta,B}$ is a space of testing functions, and its dual $\left({}^{p}S_{\mu}^{\beta,B}\right)'$, is a complete space of generalized functions, equiped either with the weak topology or with the strong topology.

If $H_{\mu}^{\beta,B}$ denotes the space defined by González[2], consisting of all the smooth functions ψ defined on $I = (0, \infty)$ such that

$$\sup_{x \in I} | x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_{m,\rho} (B + \rho)^m m^{m\beta}$$

for $m, q \in N$ and $\rho > 0$, one has :

Property 2.2.3—(a) ${}^{p}S_{\mu}^{\beta,B} \subset H_{\mu}^{p\beta, Bp^{p\beta}}$

(b) if $p > 1$, then $H_{\mu}^{\beta,B} \subset {}^{p}S_{\mu}^{\beta,B}$

(c) $H_{\mu}^{\beta,B} \subset {}^{1}S_{\mu}^{r\beta,B}$, where $r > 1$

the inclusions being continuous.

Our next result is an useful test of convergence in ${}^{p}S_{\mu}^{\beta,B}$.

Property 2.2.4—Let $\{\psi_\nu\}_{\nu \in N}$ be a sequence. If a positive constant $C_{m,\rho}$ exists for any $m \in N$, and $\rho > 0$ such that $\|\psi_\nu\|^{m,\rho} < C_{m,\rho}$ for every $\nu \in N$ and $D^q (x^{-\mu} \psi_\nu(x)) \to 0$ as $\nu \to \infty$, uniformly on $x \in (0, \epsilon)$, for every $q \in N$ and $\epsilon > 0$, then, $\psi_\nu \to 0$ as $\nu \to \infty$ in ${}^{p}S_{\mu}^{\beta,B}$.

Proof : Let $m \in N$ and $\rho, \eta > 0$. Choose ρ' such that $0 < \rho' < \rho$. In these conditions,

$$\|\psi_\nu\|^{m,\rho} < C_{m,\rho'} \neq 0, \quad \text{for every } \nu \in N$$

where $C_{m,\rho'}$ is a constant. Moreover, there exists $q_0 \in N$ such that

$$\left(\frac{B + \rho'}{B + \rho}\right)^q < \frac{\eta}{C_{m,\rho'}} \quad \text{for every } q \geqslant q_0. \text{ Hence}$$

$$| x^m D^q (x^{-\mu} \psi_\nu(x)) | < C_{m,\rho'} (B + \rho')^q (pq)!^{\beta} < \eta (B + \rho)^q (pq)!^{\beta},$$

$$\text{for } q > q_0.$$

By taking $q < q_0$ and $x > C_{m+1,\rho}/\eta$, we have

$$| x^m D^q (x^{-\mu} \psi_\nu(x)) | = \frac{| x^{m+1} D^q (x^{-\mu} \psi_\nu(x)) |}{x}$$

(equation continued on p. 588)

$$\leqslant \frac{1}{x} \|\psi_\nu\|^{m+1,\rho} (B + \rho)^q (pq)!^\beta$$

$$< \eta (B + \rho)^q (pq)!^\beta$$

and in virtue of uniform convergence, there exists a $\nu_0 \in N$ such that

$$| x^m D^q (x^{-\mu} \psi_\nu (x)) | \leqslant \eta (B + \rho)^q (pq)!^\beta,$$

for $\nu \geqslant \nu_0$, $q < q_0$ and $x < C_{m+1,\rho}/\eta$.

Therefore

$$| x^m D^q (x^{-\mu} \psi_\nu (x)) | \leqslant \eta (B + \rho)^q (pq)!^\beta$$

for $\nu \geqslant \nu_0$, $x \in I$ and $q \in N$

or in other terms: $\|\psi_\nu\|^{m,\rho} \leqslant \eta$, for $\nu \geqslant \nu_0$. Hence, $\psi_\nu \to 0$ as $\nu \to \infty$ in ${}_pS_\mu^{\beta,B}$.

Property 2.2.5—${}_pS_{\mu+k}^{\beta,B}$ is contained in ${}_pS_\mu^{\beta,B}$, and the topology of ${}_pS_{\mu+k}^{\beta,B}$ is stronger that the one induced in it by ${}_pS_\mu^{\beta,B}$, for every $k \in N$.

The proof of this property is similar to that of Property 2.1.5.

Property 2.2.6—If $0 < B_1 < B_2$ then ${}_pS_\mu^{\beta,B_1} \subset {}_pS_\mu^{\beta,B_2}$, the inclusions being continuous.

This allows to define the countably union space

$${}_pS_\mu^\beta = \bigcup_{B=1}^{\infty} {}_pS_\mu^{\beta,B}.$$

${}_pS_\mu^\beta$ equiped with the inductive limit topology is a space of testing functions and its dual $\left({}_pS_\mu^\beta \right)'$, is a space of generalized functions.

2.3. *The space* ${}_pS_{\mu,\alpha,A}^{\beta,B}$

Let μ be a real number, $\alpha, \beta \geqslant 0$ and $A, B > 0$. The space ${}_pS_{\mu,\alpha,A}^{\beta,B}$ consists of all the smooth complex valued functions ψ defined on $0 < x < \infty$ such that

$$\sup_{x \in I} | x^m D^q (x^{-\mu} \psi (x)) | < C_{\delta,\rho} (A + \delta)^m (pm)!^\alpha (B + \rho)^q (pq)!^\beta$$

for every $m, q \in N$, $\delta, \rho > 0$ with $C_{\delta,\rho}$ is a constant depending on ψ.

We consider on the space ${}_pS_{\mu,\alpha,A}^{\beta,B}$ the norms

$$\|\psi\|_{\delta}^{\rho} = \sup_{\substack{x \in I \\ m \in N \\ q \in N}} \frac{| x^m D^q (x^{-\mu} \psi(x)) |}{(A + \delta)^m (pm)!^{\alpha} (B + \rho)^q (pq)!^{\beta}} \quad \text{for all } \delta, \rho > 0.$$

The systems of norms $\Gamma_1 = \left\{ \| \ \|_{\delta}^{\rho} \right\}_{\delta, \rho > 0}$ and $\Gamma_2 = \left\{ \| \ \|_{1/n}^{1/n} \right\}_{n \in N}$ are equivalent. The space ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ endowed with the topology generated by Γ_2, is a countably multinormed space.

We now present some properties of this space that are similar to those of ${}^{p}S_{\mu,\alpha,A}$ and ${}^{p}S_{\mu}^{\beta,B}$.

Property 2.3.1—${}^{p}S_{\mu,\alpha,A}^{\beta,B} \subset H_{\mu}$ and the inclusion being continuous.

Property 2.3.2—${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ is a Fréchet space.

Therefore ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ is a space of testing functions, and its dual $\left({}^{p}S_{\mu,\alpha,A}^{\beta,B} \right)'$ is a space of generalized functions that is complete with the weak and the strong topologies.

We denote by $H_{\mu,\alpha,A}^{\beta,B}$ the space defined by Gonzàlez[2] that is constituted by the smooth complex valued functions ψ on I satisfying

$$\sup_{x \in I} | x^m D^q (x^{-\mu} \psi(x)) | \leqslant C_{\delta,\rho} (A + \delta)^m m^{m\alpha} (B + \rho)^q q^{q\beta}$$

for $m, q \in N$ and $\delta, \rho > 0$.

Property 2.3.3—(a) ${}^{p}S_{\mu,\alpha,A}^{\beta,B} \subset H_{\mu,p\alpha,Ap^{p\alpha}}^{p\beta,Bp^{p\beta}}$

(b) if $p > 1$, thin $H_{\mu,\alpha,A}^{\beta,B} \subset {}^{p}S_{\mu,\alpha,A}^{\beta,B}$

(c) $H_{\mu,\alpha,A}^{\beta,B} \subset {}^{p}S_{\mu,r\alpha,A}^{r_1 \beta,B}$, with $r > 1$ and $r_1 > 1$.

The inclusions are continuous.

Property 2.3.4—${}^{p}S_{\mu+k,\alpha,A}^{\beta,B}$ is contained in ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ and the topology of ${}^{p}S_{\mu+k,\alpha,A}^{\beta,B}$ is stronger than the topology induced in it by ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$, for every $k \in N$.

The following test of convergence can be proved as in Property 2.2.4.

Property 2.3.5—Let $\{\psi_{\nu}\}_{\nu \in N}$ be a sequence in ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$. If for each $\delta, \rho > 0$

(i) there exists a positive constant $C_{s,\rho}$ such that $\|\psi_\nu\|_s^\rho < C_{s,\rho}$, for every $\nu \in N$ and

(ii) $D^q(x^{-\mu}\psi_\nu(x))$ converges to 0, as $\nu \to \infty$, uniformly on $y \in (0, \epsilon)$ for every $q \in N$, $\epsilon > 0$

then $\psi_\nu \to 0$, as $\nu \to \infty$, in ${}^pS^{\beta,B}_{\mu,\alpha,A}$.

Property 2.3.6—If $0 < A_1 < A_2$ and $0 < B_1 < B_2$, then

$$ {}^pS^{\beta,B_1}_{\mu,\alpha,A_1} \subset {}^pS^{\beta,B_2}_{\mu,\alpha,A_2} $$

the inclusion being continuous.

We can construct the countable union space:

$$ {}^pS^{\beta}_{\mu,\alpha} = \bigcup_{\substack{A=1\\B=1}}^{\infty} {}^pS^{\beta,B}_{\mu,\alpha,A} $$

${}^pS^{\beta}_{\mu,\alpha}$ is equiped with the inductive limit topology.

2.4 The space ${}^p\hat{S}^{\beta,B}_{\mu}$:

Let μ be a real number, $\beta \geqslant 0$ and $B > 0$. We define the function space:

$$ {}^p\hat{S}^{\beta,B}_{\mu} = \left\{ \psi \in C\, {}^pS^{\beta,B}_{\mu} : C_{k,\rho} \leqslant C'_\rho \text{, for } k \in N,\ \rho > 0 \right\}. $$

This space is endowed with the topology induced in it by ${}^pS^{\beta,B}_{\mu}$.

2.5 On the Nontriviality of the Spaces of type ${}^pS^{\beta}_{\mu,\alpha}$ is related to nontriviality of spaces of type $H^{\beta}_{\mu,\alpha}$ studied by González[2].

This last author proved that the mapping

$$ S^{\beta}_{\alpha} \to H^{\beta}_{\mu,\alpha} $$

$$ \psi(y) \to y^{\mu}\psi(y) $$

is linear and continuous. The properties of spaces of type S^{β}_{α} (see Gelfand and Shilov[1]), 2.1.5, 2.2.3 and 2.3.3 show that the following spaces are nontrivial:

(a) ${}^{p}S_{\mu,\alpha,A}$ and ${}^{p}S^{\beta,B}$ for every $\alpha > 0$, $\beta \geqslant 0$, $B > 0$ and $A > 0$,

(b) ${}^{p}S^{\beta,B}_{\mu,\alpha,H}$ for $\alpha > 1$ and $\beta = 0$ or $\alpha = 0$ and $\beta > 1$; $A, B > 0$.

(c) ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$ for $A, B > 0$ and $\alpha + \beta > 1$.

(d) ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$ for $p > 1$, $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and $A. B > \gamma$, where γ is a positive constant.

3. Operational Calculus

In this section we show that derivation, multiplication by x, and some important linear differential operators, can be defined and are continuous on the previously introduced spaces.

Property 3.1—The mapping $x^n : {}^{p}S^{\beta}_{\mu,\alpha} \to {}^{p}S^{\beta}_{\mu,\alpha}$ is linear and continuous for every $n \in N$.

Proof : In effect, assuming $n = 1$ and taking, for example, ψ in ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$, then

$$\begin{aligned} |x^m D^q (x^{-\mu} x \psi(x))| &\leqslant |x^{m+1} D^q (x^{-\mu} \psi(x))| \\ &\quad + q\, |x^m D^{q-1} (x^{-\mu} \psi(x))| \\ &\leqslant C_{\delta,\rho} \{(A+\delta)^{m+1} (p(m+1))!^{\alpha} (B+\rho)^q (pq)!^{\beta} \\ &\quad + q (A+\delta)^m (pm)!^{\alpha} (B+\rho)^{q-1} (p(q-1))!^{\beta}\} \\ &\leqslant C'_{\delta,\rho} (A+\delta)^m (pm)!^{\alpha} (B+\rho)^q (pq)!^{\beta} \end{aligned}$$

for every $m \in N$, $q \in N - \{0\}$ and $\delta, \rho > 0$ in virtue of the Stirling formula and the inequality $q < (1+\epsilon)^q C_\epsilon$ where C_ϵ is a positive constant, for every $\epsilon > 0$.

Moreover $|x^m x^{-\mu} x \psi(x)| \leqslant C'_{\delta,\rho} (A+\delta)^m (pm)!^{\alpha} (B+\rho)^0 (0\,p)!^{\beta}$

for $m \in N$ and $\delta, \rho > 0$.

Hence, $x\psi$ is in ${}^{p}S^{\beta,B}_{\mu,\alpha,A}$ and the mapping is continuous.

The proof is completed by induction on n.

The procedure is analogous in any of the spaces under consideration.

Property 3.2—Let l be a real number. If we denote by ${}^{p}S^{\beta}_{\mu,\alpha}$ any of the spaces

${}^{p}S_{\mu,\alpha,A}$, ${}^{p}S_{\mu}^{\beta,B}$, ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$ or the respective union spaces, then the operator

$$x^{l} : {}^{p}S_{\mu,\alpha}^{\beta} \to {}^{p}S_{\mu+l,\alpha}^{\beta}$$

is an isomorphism.

It can be easily proved by using the definitions of these spaces.

Property 3.3—The operator $P_{\mu} = x^{\mu+1} Dx^{-\mu}$ is an isomorphism of ${}^{p}S_{\mu,\alpha}^{\beta}$ onto ${}^{p}S_{\mu+1,\alpha}^{\beta}$, and its inverse is given by

$$P_{\mu}^{-1} (\psi) (x) = x^{\mu} \int_{\infty}^{x} t^{-\mu-1} \psi (t) \, dt.$$

PROOF : Operator P_{μ} and its inverse are linear. If $\psi \in {}^{p}S_{\mu,\alpha,A}^{\beta,B}$ then

$$| x^{m} D^{q} (x^{-\mu-1} (P_{\mu} \psi) (x) | \leqslant | x^{m} D^{q+1} (x^{-\mu} \psi (x)) |$$
$$\leqslant C_{\delta,\rho} (A + \delta)^{m} (pm)!^{\alpha} (B + \rho)^{q} (pq)!^{\beta}$$

for every $m, q \in N$ and $\delta, \rho > 0$. Here too it is enough to use Stirling formula.

Hence, P_{μ} is a continuous mapping.

Moreover if $\psi \in {}^{p}S_{\mu+1,\alpha,A}^{\beta,B}$, then:

$$| x^{m} D^{q} (x^{-\mu} P_{\mu}^{-1} \psi (x)) | = | x^{m} D^{q} \left(\int_{\infty}^{x} t^{-\mu-1} \psi (t) \, dt \right) | = L (x, m, q)$$

expression that, if $q > 0$, results equal or less than

$$| x^{m} D^{q-1} (x^{-\mu-1} \psi (x)) | \leqslant C_{\delta,\rho} (A + \delta)^{m} (pm)!^{\alpha} (B + \rho)^{q} (pq)!^{\beta}$$

If $q = 0$, one has:

$$L (x, m, 0) \leqslant \int_{0}^{\infty} \frac{(t^{m+2} + t^{m}) \, (t^{-\mu-1} | \psi(t) |)}{1 + t^{2}} \, dt < C_{\delta,\rho} (A + \delta)^{m} (pm)!^{\beta}.$$

Hence $P_{\mu}^{-1} \psi$ is in ${}^{p}S_{\mu,\alpha,A}^{\beta,B}$, and P_{μ}^{-1} is a continuous operator.

The proof is similar in the case of spaces of the type ${}^{p}S_{\mu}^{\beta,B}$ or ${}^{p}S_{\mu,\alpha,A}$.

Property 3.4—The mapping $D : {}^{p}S^{\beta}_{\mu,\alpha} \rightarrow {}^{p}S^{\beta}_{\mu-1,\alpha}$ is linear and continuous.

PROOF : We limit to ourselves to the case of the operator

$$D : {}^{p}S^{\beta,B}_{\mu,\alpha,A} \rightarrow {}^{p}S^{\beta,B}_{\mu-1,\alpha,A}$$

since the other cases can be deduced from this one.

Let $\psi \in {}^{p}S^{\beta,B}_{\mu,\alpha,A}$. Then,

$$\begin{aligned} | x^m D^q (x^{-\mu+1} \psi(x)) | &\leqslant | x^m D^q (x^{-\mu} \psi(x)) (\mu + q) | \\ &\quad + | x^{m+1} D^{q+1} (x^{-\mu} \psi(x)) | \\ &\leqslant C_{\delta,\rho} (A + \delta)^m (pm)!^{\alpha} (B + \rho) (pq)!^{\beta} \end{aligned}$$

for every $m, q \in N$ and $\delta, \rho > 0$, following a procedure similar to the one used above.

From the previous results it can be easily inferred.

Property 3.5—The operator $B_\mu = DP_\mu$ from ${}^{p}S^{\beta}_{\mu,\alpha}$ into itself is linear and continuous.

Defining the generalized D^*, P^*_μ, P^{-1*}_μ and B^*_μ as the adjoint of the classical operators D, P_μ, P^{-1}_μ and B_μ respectively the following:

Property 3.6—The operators $D^* : \left({}^{p}S^{\beta}_{\mu-1,\alpha} \right)' \rightarrow \left({}^{p}S^{\beta}_{\mu,\alpha} \right)'$ and $B^*_\mu : \left({}^{p}S^{\beta}_{\mu,\alpha} \right)' \rightarrow \left({}^{p}S^{\beta}_{\mu,\alpha} \right)'$ are linear and continuous.

The mapping $P^*_\mu : \left({}^{p}S^{\beta}_{\mu+1,\alpha} \right)' \rightarrow \left({}^{p}S^{\beta}_{\mu,\alpha} \right)'$ is an isomorphism P^{-1*}_μ is its inverse.

4. MULTIPLIERS IN SPACES OF TYPE ${}^{p}S^{\beta}_{\mu,\alpha}$

We are now interested in smooth functions on $0 < x < \infty$ which are multipliers in spaces of type ${}^{p}S^{\beta}_{\mu,\alpha}$.

Let $\theta \in C^{\infty}(I)$ be a function such that:

$$| D^q \theta(x) | \leqslant C\, B_0{}^q (pq)!^{\beta} (1 + x^l)$$

where $l \in N$ and $C, B_0 > 0$. Also, let $\psi \in {}^pS_{\mu,\alpha,A}$. Hence

$$| x^m D^n (x^{-\mu} \theta(x)\psi(x)) | \leqslant C_{n,\delta} \sum_{r=0}^{n} \binom{n}{r} B_0{}^r (pr)!^\beta (A+\delta)_m (pm)!^\alpha$$

$$\times (1 + (A+\delta) \frac{(p(m+l))!^\alpha}{(pm)!^\alpha}$$

$$< C'_{n,\delta} (A+\delta)^m (pm)!^\alpha$$

for $m, n \in N$ and $\delta > 0$.

Thus $\theta\psi$ is in ${}^pS_{\mu,\alpha,A}$ and the mapping

${}^pS_{\mu,\alpha,A} \to {}^pS_{\mu,\alpha,A}$

$\psi \to \theta\psi$ is continuous.

Moreover taking a ψ in ${}^pS_\mu^{\beta,B}$

$$| x^m D^n (x^{-\mu} \psi(x) \theta(x)) | \leqslant C_{m,\rho} (pm)!^\beta \sum_{r=0}^{n} \binom{n}{r} B_0{}^r (B+\rho)^{n-r}$$

$$= C_{m,\rho} (pm)!^\beta (B + B_0 + \rho)^n$$

for $m, n \in N$ and $\rho > 0$, since $(pr)! \cdot (p(n-r))! < (pn)!$. Therefore $\theta\psi$ is in ${}^pS_\mu^{\beta,B+B_0}$ and the operator

${}^pS_\mu^{\beta,B} \to {}^pS_\mu^{\beta,B+B_0}$

$\psi \to \theta\psi$ is continuous.

If $\psi \in {}^pS_{\mu,\alpha,A}^{\beta,B}$ then

$$| x^m D^n (x^{-\mu} \psi(x) \theta(x)) | < C_{\delta,\rho} (A+\rho)^m (pm)!^\alpha (pn)!^\beta (B + B_0 + \rho)^n$$

for every $m, n \in N$ and $\delta, \rho > 0$. Hence $\psi\theta$ is in ${}^pS_{\mu,\alpha,A}^{\beta,B+B_0}$, and the mapping

${}^pS_{\mu,\alpha,A}^{\beta,B} \to {}^pS_{\mu,\alpha,A}^{\beta,B+B_0}$

$\psi \to \theta\psi$ is continuous.

These facts are summarized in the following:

Property 4.1—If $\theta \in C^\infty(I)$ and $| D^q \theta(x) | \leqslant C B_0{}^q (pq)!^\beta (1 + x^l)$ for every $q \in N$, with $l \in N$ and $B_0, C > 0$, then θ is a multiplier of

(a) ${}_pS_{\mu,\alpha,A}$ into itself (and of $({}^pS_{\mu,\alpha,A})'$ into itself),

(b) ${}^pS_{\mu}^{\beta,B}$ into ${}^pS_{\mu}^{\beta, B+B_0}$ $\left(\text{and of } \left({}^pS_{\mu}^{\beta, B+B_0} \right)' \text{ into } \left({}^pS_{\mu}^{\beta,B} \right)' \right)$,

(c) ${}^pS_{\mu,\alpha,A}^{\beta,B}$ into ${}^pS_{\mu,\alpha,A}^{\beta, B+B_0}$ $\left(\text{and of } \left({}^pS_{\mu,\alpha,A}^{\beta, B+B_0} \right)' \text{ into } \left({}^p_cS_{\mu,\alpha,A}^{\beta,B} \right)' \right)$.

This result can be extended to the respective union spaces.

We now consider the set M, defined by Méndez[5], consisting of the smooth functions on $0 < x < \infty$ such that for each $r \in N$ there exists a $n_r \in N$ for which the function

$$\frac{D^r \theta(x)}{1 + x^{n_r}}$$

is bounded on $0 < x < \infty$. M is a space of multipliers in H_μ.

Property 4.2—If $\theta \in M$, the mapping

$${}^pS_{\mu,\alpha,A} \to {}^pS_{\mu,\alpha,A}$$

$\psi \to \psi\theta$ is linear and continuous.

Proof : It is enough to check that

$$| x^m D^q (x^{-\mu} \theta(x) \psi(x)) | \leqslant \sum_{r=0}^{q} \binom{q}{r} \frac{| D^r \theta(x) |}{1 + x^{n_r}} (x^m + x^{m+n_r})$$

$$\times \; | D^{q-r} (x^{-\mu} \psi(x)) | \leqslant C_{q,s} (A+\delta)^m (pm)!^\alpha$$

for each $\psi \in {}^pS_{\mu,\alpha,A}$, $m, q \in N$, and suitable nonnegative integers n_r.

This result can be extended to the union space ${}^pS_{\mu,\alpha}^{\beta}$ and the respective dual spaces.

Note that a similar result cannot always be extended for ${}^pS_{\mu}^{\beta,B}$ and ${}^pS_{\mu,\alpha,A}^{\beta,B}$.

5. The Hankel-Clifford Transformation in the Spaces of Type ${}^pS_{\mu,\alpha}^{\beta}$

An integral transform given by the pair

$$F(y) = h_\mu \{f(x)\}(y) = y^\mu \int_0^\infty C_\mu(xy) f(x)\, dx$$

$$f(x) = h_\mu \{F(y)\}(x) = x^\mu \int_0^\infty C_\mu(xy) F(y)\, dy \qquad ...(1)$$

was defined by Méndez[5], who called it the Hankel-Clifford transformation. The kernel of this transform, C_μ, is the Bessel-Clifford function of the first kind and order μ. C_μ is related to the Bessel function J_μ of the first kind by $C_\mu(z) = z^{-\mu/2} J_\mu(2\sqrt{z})$ (see Hayek[3]).

This transformation is an automorphism onto H_μ, for $\mu \geqslant 0$.

Since $\frac{d^n}{dz^n} C_\mu(z) = (-1)^n C_{\mu+n}(z)$, $n \in N$, then for every $\phi \in H_\mu$ and $m, k \in N$

$$y^m D^k y^{-\mu} \psi(y) = (-1)^k \int_0^\infty C_{\mu+k+2m}(xy)(xy)^m x^{\mu+k+m} D^{2m} x^{-\mu} \phi(x)\, dx$$

where $\psi(y) = h_\mu\{\phi(x)\}(y)$.

By dividing the integral $\int_0^\infty = \int_0^1 + \int_1^\infty$ and in virtue of the boundedness of the function $z^m C_{\mu+k+2m}(z)$, we obtain

$$|y^m D^k (y^{-\mu} \psi(y))| \leqslant M \{\sup_{x \in I} |x^{c+k+m} D^{2m} x^{-\mu} \phi(x)| + \sup_{x \in I} |x^{c+k+m+2} D^{2m} x^{-\mu} \phi(x)|\} \quad \ldots(2)$$

for $m, k \in N$ and $\mu \geqslant 0$, being $c = [\mu]$ and M a constant.

We now study the image of ${}^pS^\beta_{\mu,\alpha}$ by h_μ. Recall that these spaces are contained in H_μ.

Let ϕ be an element of ${}_pS_{\mu,\alpha,A}$, invoking (2) we obtain

$$|y^m D^k (y^{-\mu} \psi(y))| \leqslant K_{m,\delta}(A + \delta)^k \{(p(m + k + c))!^\alpha + (p(c + k + m + 2))!^\alpha\} \leqslant C_{m,\delta}(A + \delta)^k (pk)!^\alpha$$

for every $m, k \in N$ and $\delta > 0$.

Hence, the mapping $h_\mu : {}^pS_{\mu,\alpha,A} \to {}^pS^{\alpha,A}_\mu$ is linear and continuous.

If $\phi \in {}^p\hat{S}^{\beta,B}_\mu$, then

$$|y^m D^k y^{-\mu} \psi(y)| \leqslant M\{C_{k+m+c,\rho}(B + \rho)^{2m}(2mp)!^\beta + C_{k+m+2+c,\rho}(B + \rho)^{2m}(2mp)!^\beta\}.$$

Since $C_{k+m+c,\rho} < C'_{k,\rho}$ for every $m \in N$

$$|y^m D^k y^{-\mu} \psi(y)| \leqslant M_{k,\eta}(B^2 + \eta)^m (2mp)!^{2\beta}$$

for $m, k \in N$ and $\eta > 0$.

Also, from the Stirling formula the next equality follows:

$$| y^m D^k y^{-\mu} \psi (y) | \leqslant M_{k,\eta} (B^2 2^{2p} + \eta)^m (pm)!^{2\beta}$$

for $m, k \in N$ and $\eta > 0$.

Therefore the mappings:

$$h_\mu : {}^p\hat{S}_\mu^{\beta, B} \rightarrow {}^{2p}S_{\mu,\beta,B^2}$$

$$h_\mu : {}^p\hat{S}_\mu^{\beta, B} \rightarrow {}^pS_{\mu,2\beta,2^{2p\beta}B^2}$$

are linear and continuous.

Let now ϕ be in ${}^pS^{\beta,B}_{\mu,\alpha,A}$. According to (2), we have

$$| y^m D^k y^{-\mu} \psi (y) | \leqslant M_{\delta,\rho} \{(A + \delta)^{m+k+c} (p (c + k + m))!^\alpha (B + \rho)^{2m} \times (2pm)!^\beta + (A + \delta)^{c+k+m+2} (p (c+k+m+2))!^\alpha \times (B + \rho)!^{2m} (2pm)!^\beta\} \leqslant C_{\eta,\epsilon} (Ae p^\alpha + \eta)^k \times (pm)!^{2\beta+\alpha} (AB^2 2^{2p\beta} e p^\alpha + \epsilon)^m (pk)!^\alpha$$

for every $m, k \in N$ and $\eta, \epsilon > 0$.

Thus, it has been established that the mapping

$$h_\mu : {}^pS^{\beta,B}_{\mu,\alpha,A} \rightarrow {}^pS^{\alpha, Ae^{p\alpha}}_{\mu,2\beta+\alpha,B\,2A2\;2^{p}\beta e\;p\alpha}$$

is linear and continuous.

The above results are summarized in the next

Theorem 5.1—The mappings

$$h_\mu : {}^pS^{\beta,B}_{\mu,\alpha,A} \rightarrow {}^pS^{\alpha, Ae^{p\alpha}}_{\mu,2\beta+\alpha,AB^2 2^{2p\beta}e\,p\alpha}$$

$$h_\mu : {}^p\hat{S}_\mu^{\beta, B} \rightarrow {}^{2p}S_{\mu,\beta,B^2}$$

$$h_\mu : {}^p\hat{S}_\mu^{\beta, B} \rightarrow {}^pS_{\mu,2\beta,2^{2p\beta}B^2}$$

$$h_\mu : {}^pS_{\mu,\alpha,A} \rightarrow {}^pS_\mu^{\alpha, A}$$

are linear and continuous.

Defining the generalized transformation h'_μ as the adjoint of the classical transformation h_μ, it can easily be seen that

Theorem 5.2—The operators

$$h'_\mu : \left({}^{p}S^{\alpha, Ae^{p\alpha}}_{\mu, 2\beta+\alpha, AB^2\, 2_2\, 2p\beta e\, p\alpha} \right)' \to \left({}^{p}S^{\beta, B}_{\mu, \alpha, A} \right)'$$

$$h'_\mu : ({}^{p}S_{\mu, 2\beta, 2\; 2p\beta_B\, 2})' \to \left({}^{p}\hat{S}^{\beta, B}_{\mu} \right)'$$

$$h'_\mu : ({}^{2p}S_{\mu, \beta, B\, 2})' \to \left({}^{p}\hat{S}^{\beta, B}_{\mu} \right)'$$

$$h'_\mu : \left({}^{p}S^{\alpha, A}_{\mu} \right)' \to ({}^{p}S_{\mu, \alpha, A})'$$

are linear and continuous.

6. A Cauchy Problem Containing the Bessel-type Operator $B_\mu = Dx^{\mu+1}\, Dx^{-\mu}$

We consider the Cauchy problem

$$\left.\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= P(B_\mu)\, u(x, t) \\ u(x, t_0) &= \phi_0(x) \end{aligned}\right\} \qquad \ldots(3)$$

for $0 \leqslant t \leqslant t_0 \leqslant T$, with $u(x, t)$ an unknown vector function and P a square matrix of polynomials. The initial value ϕ_0 is in a fundamental space that will be determined later.

Application of the h_μ-transform to problem (3) leads to

$$\left.\begin{aligned} \frac{\partial v(y, t)}{\partial t} &= P(-y)\, v(y, t) \\ v(y, t_0) &= \psi_0(y) \end{aligned}\right\} \qquad \ldots(4)$$

where $v(y, t) = h_\mu \{u(x, t)\, ;\, x \to y\}$ and $\psi_0(y) = h_\mu \{\psi_0(x)\}(y)$. A formal solution of (4) is

$$v(y, t) = Q(-y, t, t_0)\, \psi_0(y)$$

where

$$Q(-y, t, t_0) = \exp((t - t_0)\, P(-y)).$$

A bound for the matrix $Q(s, t, t_0)$ is given by (see Geland and Shilov[1], pp. 53)

$$\| Q(s, t, t_0) \| \leqslant C \exp\left(\left(\frac{\theta}{2}\right)^{p_0} \frac{1}{p_0} |s|^{p_0} \right)$$

for $t \in [0, T]$ (even $t \in [0, T + t_0]$), $2^{p_0+1}\, p_0 T < \frac{1}{p_0}\, \theta^{p_0}$ and p_0 being the reduced order of the system (3) when the operator B_μ is substituted by $i \frac{\partial}{\partial x}$.

In virtue of Cauchy's integral formula

$$D^q Q(-y, t, t_0) = \frac{q}{2\pi i} \int_0^{2\pi} \frac{Q(-y + Re^{i\alpha}, t, t_0)\, Re^{i\alpha}\, i}{(Re^{i\alpha})^{q+1}}\, d\alpha.$$

Hence

$$\| D^q Q(-y, t, t_0) \| \leqslant q!\, R^{-q}\, C \exp\left(\frac{(2\theta)^{p_0}}{p_0}\, y^{p_0}\right) \exp\left(\frac{(2\theta)^{p_0} R^{p_0}}{p_0}\right)$$

where $C > 0$.

If we assume $R = q^{1/p_0}$, then

$$\| D^q Q(-y, t, t_0) \| \leqslant C_1\, q^{q\left(a - \frac{1}{q_0}\right)} \exp\left(\frac{(2\theta)^{p_0}}{p_0}\, y^{p_0}\right) \times \left(\exp\left(\frac{(2\theta)^{p_0}}{p_0}\right)\right)^q$$

for $a \geqslant 1$.

And we arrive at the following

Property 6.1—The matrix $Q(-y, t, t_0)$ is a multiplier of the space

$${}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right),\, B}_{\mu,\, \frac{1}{pp_0},\, A} \quad \text{in} \quad {}^pS^{a - \frac{1}{p_0},\, Bp^{\left(a - \frac{1}{p_0}\right)} + B_0}_{\mu,\, \frac{1}{p_0},\, Ap^{1/p_0} + A_0} \qquad \text{for } p > 1$$

where

$$B_0 = \exp\left((2\theta)^{p_0}/p_0\right)$$

$$A_0 = \left(\frac{1}{pp_0\, e\, (c - a_0)}\right)^\alpha$$

with

$$a_0 = \frac{(2\theta)^{p_0}}{p_0} \quad c = \frac{\alpha}{e} A^{-1/\alpha} \quad \alpha = \frac{1}{pp_0}.$$

If $p = 1$ then $Q(-y, t, t_0)$ is a multiplier of

$${}^1S^{a - \frac{1}{p_0},\, B}_{\mu,\, \frac{1}{p_0},\, A} \quad \text{in} \quad {}^1S^{\epsilon\left(a - \frac{1}{p_0}\right),\, B + B_0}_{\mu,\, \frac{\epsilon}{p},\, A + A_0}$$

with $\epsilon > 1$.

This property has sense provided that the space ${}^pS^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$ under consideration is nontrivial.

By invoking 1.5 we get

Property 6.2—The space ${}^pS^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$ is nontrivial if

(a) $p_0 > \frac{1}{a}$ and $a > p \geqslant 1$

(b) $p_0 > \frac{1}{a}$, $a = p > 1$ and $A \cdot B > \gamma$, γ being a positive constant

(c) $ap_0 = 1$ and $\frac{1}{pp_0} > 1$.

We now define the fundamental space to ϕ_0 (the data of the initial value problem (3)) belongs to. According to Theorem 1, if $\Phi = {}^pS^{\beta,B'}_{\mu,\alpha,A'}$ and

$$h_\mu\{\Phi\} \subseteq {}^pS^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$$

(provided that ${}^pS^{\frac{1}{p}\left(a-\frac{1}{p_0}\right),B}_{\mu,\frac{1}{pp_0},A}$ is nontrivial) then:

$$\beta = \frac{1}{pp_0} - \frac{a}{2p}, \quad \alpha = \frac{1}{p}\left(a - \frac{1}{p_0}\right) \text{ and } A', B' > 0 \text{ with}$$

$$A'\, B'\, 2^{2p\beta}\, e^{p\alpha} = A, \quad A'\, e^{p\alpha} = B.$$

The space Φ is nontrivial if some of these conditions hold

(a) $\frac{a}{2p} > 1$ and $a > \frac{1}{p_0} > \frac{a}{2}$

(b) $a = 2p$, $a > \frac{1}{p_0} > \frac{a}{2}$ and $A'\, B' > \gamma$, where γ is a positive constant

(c) Either $a = \frac{1}{p}$ and $1 > 2pp_0$, or $ap_0 = 2$ and $1 > pp_0$.

When the space Φ is trivial a new

$$\Phi = \{\psi \in H_\mu : {}_ch_\mu\, \psi \in {}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right), B}_{\mu, \frac{1}{pp_0}, A}\}$$

has to be introduced.

In this case Φ is endowed with the graph topology. This topology is generated by the multinorm $\left\{ | \ | {}^{\rho}_{\delta} \right\}_{\delta, \rho > 0}$ where

$$| \psi | {}^{\rho}_{\delta} = \| {}_ch_\mu\, \psi \| {}^{\rho}_{\delta} \quad \text{for } \psi \in \Phi \text{ and } \delta, \rho > 0$$

and where the last seminorm is defined in the space

$${}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right), B}_{\mu, \frac{1}{pp_0}, A}.$$

The nontriviality of Φ is deduced from the nontriviality of

$${}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right), B}_{\mu, \frac{1}{pp_0}, A}.$$

The following diagram summarizes our last results

$$\Phi \to \Phi_1 = {}^pS^{\frac{1}{p}\left(a - \frac{1}{p_0}\right), B}_{\mu, \frac{1}{pp_0}, A} \to \Phi_2 = {}^pS^{a - \frac{1}{p_0}, Bp^{p\beta} + B_0}_{\mu, \frac{1}{p_0}, Ap^{p\alpha} + A_0}$$

$$\phi_0 \to h_\mu \{\phi_0\} = \psi_0 \to v(y, t) = Q(-y, t, t_0)\, \psi_0(y).$$

Note that Φ_1 is contained in Φ_2 and the inclusion is continuous.

We now show that $v(y, t)$ is the solution of the problem (3).

Property 6.3—If $\Phi_0 \in \Phi$, then

$\lim\limits_{t \to t_0} v(y,t) = \psi_0(y)$ and $\lim\limits_{\Delta t \to 0} \dfrac{\Delta v(y, t)}{\Delta t} = P(-y)\, v(y, t)$ in the sense of the convergence in Φ_2.

Hence, in virtue of the operational rules in Méndez[5] and taking into account that ${}_ch_\mu$ is a continuous mapping, we get

$$\lim_{t \to t_0} h_\mu \{v(y, t), y \to x\} = h_\mu \{\psi_0(y)\}(x) = \phi_0(x)$$

$$\lim_{\Delta t \to 0} \frac{\Delta\, h_\mu \{v(y, t), y \to x\}}{\Delta t} = P(B_\mu)\, h_\mu \{v(y, t), y \to x\}$$

in the sense of convergence in the space

$$\Phi_4 = {}_{\nu}S^{\frac{1}{p_0},\,(Ap^{p\alpha}+A_0)\,e^{p/p_0}}_{\mu,\,2a-\frac{1}{p_0},\,(Ap^{p\alpha}+A_0)\,(Bp^{p\beta}+B_0)^2\,2^{2p\left(a-\frac{1}{p_0}\right)}\,e^{p/p_0}}.$$

Thus, the next result follows.

Theorem 6.1—The function $u(x, t) = h_\mu \{Q(-y, t, t_0)\, h_\mu \{\phi_0(x)\}(y), y \to x\}$ in Φ_4 is a solution of the Cauchy problem (3), for every initial value $\phi_0 \in \Phi$.

7. Existence of Generalized Solutions

We now consider the initial value problem

$$\frac{\partial u(x, t)}{\partial t} = P(B_\mu^*)\, u(x, t)$$
$$u(x, 0) = u_0(x) \text{ with } u_0 \in \Phi_4' \qquad \ldots(5)$$

The generalized Hankel-Clifford transform h'_μ, leads to the new equivalent problem

$$\frac{\partial v(y, t)}{\partial t} = P(-y)\, v(y, t)$$
$$v(y, 0) = v_0(y) \qquad \ldots(6)$$

where $v(y, t) = h'_\mu \{u(x, t), x \to y\}$ and $v_0(y) = h'_\mu \{u_0(x)\}(t)$.

A formal solution of (6) is the generalized function $v(y, t) = v_0(y)\, e^{-vt}$.

The distribution $u(x, t) = h'_\mu \{v_0(y)\, e^{-vt}, y \to x\} \in \Phi'$ is a solution of (5).

In effect, according to Theorem 6.1 one has:

(a) $$\frac{\partial}{\partial t} < h'_\mu \{e^{-vt}\, v_0(y)\}, \phi > = \frac{\partial}{\partial t} < u_0, h_\mu \{e^{-vt}\, h_\mu \{\phi\}\} >$$
$$= < u_0, h_\mu \{e^{-vt}\, h_\mu \{P(B_\mu)\}\} >$$
$$= < P\left(B_\mu^*\right) h'_\mu\, e^{-vt}\, h'_\mu\, u_0, \phi >$$

for every $\phi \in \Phi$ and

(b) $$< h'_\mu \{e^{-vt}\, h'_\mu \{u_0\}\}, \phi > = < u_0, h_\mu \{e^{-vt}\, h_\mu \{\phi\}\} > \to < u_0, \phi >$$

for every $\phi \in \Phi$.

Thus, we arrive at the following

Theorem 7.1—The generalized function

$$u\ (x, t) = h'_{\mu}\ \{e^{-yt}\ h'_{\mu}\ \{u_0\}\} \in \Phi'$$

is solution of (5) for every initial value $u_0 \in \Phi'_4$.

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