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# A FINITE INTEGRAL INVOLVING A GENERAL CLASS OF POLYNOMIALS AND THE MULTIVARIABLE *H*-FUNCTION

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In this paper, we evaluate a finite integral involving the product of a general class of polynomials and the multivariable *H*-function. On account of the most general nature of the polynomials and multivariable *H*-function, our result provide interesting unifications and extensions of a number of (known and new) integrals. Integrals obtained by Gupta *et al.*[2] (p. 69), Garg[1] (p. 251) and many other integrals follow as particular cases of our main result.

## 1. INTRODUCTION

The multivariable *H*-function has been defined by Srivastava and Panda[6]. We shall use the following contracted form Srivastava *et al.*[5] [p. 251 eqn. (C. 1)]:

$$H[z_1, \ldots, z_r] = H^{O,\, N\, :\, M',\, N';\, \ldots;\, M^{(r)},\, N^{(r)}}_{P,\, Q\, :\, P',\, Q';\, \ldots;\, P^{(r)},\, Q^{(r)}} \left[ \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} \left(a_j;\, \alpha'_j, \ldots, \alpha^{(r)}_j\right)_{1,P} : \left(c'_j, \gamma'_j\right)_{1,P'}; \ldots; \left(c^{(r)}_j, \gamma^{(r)}_j\right)_{1,P^{(r)}} \\ \left(b_j;\, \beta'_j, \ldots, \beta^{(r)}_j\right)_{1,Q} : \left(d'_j, \delta'_j\right)_{1,Q'}; \ldots; \left(d^{(r)}_j, \delta^{(r)}_j\right)_{1,Q^{(r)}} \end{matrix} \right] \quad \ldots(1.1)$$

to denote the *H*-function of $r$ complex variables $z_1, \ldots, z_r$. All the Greek letters are assumed to be positive real numbers for standardization purposes; the definition of the multivariable *H*-function will, however, be meaningful even if some of these quantities are zero. The details of this function can be found in the paper referred to above.

Srivastava[3] introduced the general class of polynomials (see also Srivastava[4] and Srivastava and Singh[7]) :

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k,\ n = 0, 1, 2, \ldots \qquad \ldots(1.2)$$

where $m$ is an arbitrary positive integer and the coefficients $A_{n,k}\,(n, k \geqslant 0)$ are arbitrary constants real or complex. By suitably specializing the coefficients $A_{n,k}$, the general class of polynomials can be reduced to a large spectrum of polynomials as cited in the papers referred to above.

## 2. Result Required

The following result given by Garg[1] (p. 244) will be required in establishing our main integral in the next section :

$$\int_0^a x^{\rho-1} (a + x)^{\sigma} (c + bx)^{-\lambda} H[z_1 x^{u_1} (a - x)^{v_1} (c + bx)^{-w_1}, \ldots,$$

$$z_r x^{u_r} (a - x)^{v_r} (c + bx)^{-w_r}]\, dx$$

$$= \sum_{l=0}^{\infty} \frac{(-ab)^l a^{\rho+\sigma}}{c^{\lambda+l}\, l!} H^{O,N+3\ :\ *}_{P+3,Q+2\ :\ *} \left[ \begin{matrix} z_1 c^{-w_1} a^{u_1+v_1} \\ \vdots \\ z_1 c^{-w_r} a^{u_r+v_r} \end{matrix} \middle| \begin{matrix} A : * \\ \\ B : * \end{matrix} \right] \qquad \ldots(2.1)$$

where $H[z_1, \ldots, z_r]$ denotes the multivariable $H$-function defined by (1.1). The asterisk ( $*$ ) occurring in (2.1) indicates that parameters at those places are the same as the parameters of the multivariable $H$-function in (1.1) at corresponding places. This notation will be adhered to throughout this paper. Also

$$A = (-\sigma; v_1, \ldots, v_r),\ (1 - \rho - l; u_1, \ldots, u_r),\ (1 - \lambda - l;\ w_1, \ldots, w_r),$$

$$\left( a_j;\ \alpha_j', \ldots, \alpha_j^{(r)} \right)_{1,P}$$

$$B = \left( b_j;\ \beta_j', \ldots, \beta_j^{(r)} \right)_{1,Q},\ (1 - \lambda; w_1, \ldots, w_r),$$

$$(-\rho - \sigma - l;\ u_1 + v_1, \ldots, u_r + v_r).$$

The conditions of validity of (2.1) can be found in the reference given above.

## 3. Main Integral

We shall establish the following finite integral in this section :

$$\int_0^a x^{\rho-1} (a - x)^{\sigma} (c + bx)^{-\lambda} S_n^m [yx^u (a - x)^v (c + bx)^{-w}]$$

$$\times\ S_{n'}^{m'} [zx^{u'} (a - x)^{v'} (c + bx)^{-w'}]$$

(*equation continued on p.* 606)

$$\times H[z_1 x^{u_1} (a-x)^{v_1} (c+bx)^{-w_1}, \ldots, z_r x^{u_r} (a-x)^{v_r} \times (c+bx)^{-w_r}]\, dx$$

$$= \sum_{l=0}^{\infty} \sum_{k=0}^{[n/m]} \sum_{k=0}^{[n'/m']} \frac{(-1)^l (-n)_{mk} (-n')_{m'k'} y^k z^{k'} a^{\rho+\sigma+k(u+v)+k'(u'+v')+l} b^l}{l!\, k!\, k'!\, c^{\lambda+wk+w'k'+l}}$$

$$\times A_{n,k} A'_{n',k'} H^{O,N+3\,:\,*}_{P+3,Q+2\,:\,*}\left[\begin{matrix} z_1\, c^{-w_1} a^{u_1+v_1} \\ \vdots \\ z_r\, c^{-w_r} a^{u_r+v_r} \end{matrix}\middle| \begin{matrix} C:* \\ \\ D:* \end{matrix}\right] \quad \text{.. (3.1)}$$

where

$$C = (-\sigma - vk - v'k'; v_1, \ldots, v_r), (1-\rho-l-uk-u'k'; u_1, \ldots, u_r)$$
$$(1-\lambda-l-wk-w'k'; w_1, \ldots, w_r), \left(a_j; \alpha'_j, \ldots, \alpha_j^{(r)}\right)_{1,P}$$

$$D = \left(b_j; \beta'_j, \ldots, \beta_j^{(r)}\right)_{1,Q}, (1-\lambda-wk-w'k'; w_1, \ldots, w_r)$$
$$(-\rho-\sigma-l-(u+v)k-(u'+v')k'; u_1+v_1, \ldots, u_r+v_r)$$

$m$ and $m'$ are arbitrary positive integers and the coefficients $A_{n,k}$ $(n, k \geqslant 0)$ and $A'_{n',k'}$ $(n', k' \geqslant 0)$ are arbitrary constants real or complex.

The (sufficient) conditions of validity of (3.1) are :

(i) $a, b, c$ are positive numbers such that $\left|\frac{ab}{c}\right| < 1$

Re $(\lambda) > 0$, $\min_{1\leqslant i\leqslant r} (u, v, w, u', v', w', u_i, v_i, w_i) \geqslant 0$ (not all zero simultaneously)

(ii) $$\text{Re}(\rho) + \sum_{i=1}^{r} u_i \min_{1\leqslant j\leqslant M} (i) \left\{ \text{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\} > 0$$

$$\text{Re}(\sigma) + \sum_{i=1}^{r} v_i \min_{1\leqslant j\leqslant M} (i) \left\{ \text{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\} + 1 > 0$$

(iii) $\Omega_i > 0$, $|\arg z_i| < \frac{1}{2}\Omega_i \pi$, $\forall\, i \in \{1, 2, \ldots, r\}$, where

$$\Omega_i = -\sum_{j=N+1}^{P} \alpha_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{M^{(i)}} \delta_j^{(i)} - \sum_{j=M^{(i)}+1}^{Q^{(i)}} \delta_j^{(i)}$$

(*equation continued on p.* 607)

$$+ \sum_{j=1}^{N^{(i)}} \gamma_j^{(i)} - \sum_{j=N^{(i)}+1}^{P^{(i)}} \gamma_j^{(i)}.$$

(iv) The series occurring on right-hand side of (3.1) is absolutely convergent.

PROOF : To evaluate the integral (3.1), we first express both the general class of polynomials occurring in the integrand of (3.1) in series from given by (1.2) and then interchange the orders of summations and integration (which is permissible under the conditions stated above) so that the left-hand side of the integral (3.1) (say $\Delta$) assumes the following form:

$$\Delta = \sum_{k=0}^{[n/m]} \sum_{k'=0}^{[n'/m']} \frac{(-n)_{mk}\,(-n')_{m'k'}\, y^k\, z^{k'}}{k!\, k'!} A_{n,k}\, A'_{n',k'} \int_0^a x^{\rho+uk+u'k'-1}$$

$$\times (a-x)^{\sigma+vk+v'k'} (c+bx)^{-\lambda-wk-w'k'}\, H[z_1 x^{u_1} (a-x)^{v_1}$$

$$\times (c+bx)^{-w_1}, \ldots, z_r x^{u_r} (a-x)^{v_r} (c+bx)^{-w_r}]\, dx. \qquad \ldots(3.2)$$

Now, we evaluate the $x$-integral occurring in the above equation, with the help of (2.1) and arrive at the desired result (3.1) after a little simplification.

## 4. SPECIAL CASES AND APPLICATIONS

If we take $n' = 0$ (the polynomial $S_0^{m'}[x]$ will reduce to 1), $\lambda = w = 0$, $w_i = 0$ $(i = 1, 2, \ldots, r)$, $b \to 0$ and replace $\rho$ by $\rho + 1$ in (3.1), we get a known integral obtained by Gupta *et al.*[2] [p. 69, eqn. (3.1)]. Further, if we take $u = 0$ and $v = 1$ in the integral so obtained, we get an integral given by Srivastava and Singh[7] [p. 166, eqn. (2.2), with $\xi = 0$].

Again, if we take $n' = 0$, $a = u = y = 1$, $v = w = 0$, $v_i = 0$ $(i = 1, 2, \ldots, r)$, $m = 1$, $A_{n,k} = \binom{n+\alpha}{n} \frac{(\alpha+\beta+n+1)_k}{(\alpha+1)_k}$ $\Big($ in this case $S_n^1[x] \to P_n^{(\alpha,\beta)}(1-2x)\Big)$ and replace $\sigma$ by $\beta$ in (3.1), we get another known integral given by Garg[1] [p. 251, eqn. (5.3.6)].

The importance of our main integral lies in its manifold generality. At the outset, we recall that in view of the generality of the polynomials $S_n^m[x]$, on suitably specializing the coefficients $A_{n,k}\, A'_{n',k'}$ and making a free use of the special cases of $S_n^m[x]$ listed by Srivastava and Singh[7], our main integral can be reduced to a large number of integrals involving generalized Hermite polynomials, Hermite polynomials, Jacobi polynomials and its various special cases, Laguerre polynomials, Bessel polynomials, Gould-Hopper polynomials, Brafman polynomials, and their various combinations.

Secondly, by specializing the various parameters and variables in the multivariable $H$-function, we can obtain from our main result, several integrals involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of $E$, $F$, $G$ and $H$-functions of one and several variables. Thus our main integral would at once yield a very large number of integrals involving a large variety of polynomials and various special functions occurring in the problems of mathematical analysis, applied mathematics, and mathematical physics.

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