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ON STRONGLY RARE-CONTINUITY

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A new weakly form of continuity which is weaker than weak-continuity and stronger than rare-continuity is introduced under the title of strongly-rare-continuity. Some certain characteristics of strongly rarely continuous functions, some of their properties in terms of α-topologies of Njastad and some of their relations with the other weakly types of functions are established.

INTRODUCTION

Many weak forms of continuity of single valued functions between two topological spaces has been defined in the literature since 1920's. Quasi-continuities of Blumberg[3] and Kempisty[8] were introduced in 1922 and 1932 in two different ways. Fomin[5] have defined the θ-continuity in 1941 and Levine have defined weak-continuity [9] and semi-continuity[10] in 1961 and 1963 respectively. Two independent kinds of almost continuity were defined by Husain[7] in 1966 and by Singal and Singal[19] in 1968. Faint-continuity of Long and Herrington[11], rare-continuity of Popa[16] and subweak-continuity of Rose[17] have been defined in the last decade for instance δ-continuity of Noiri which is an independent concept of continuity have also been defined in this period. The following non reversible implications among some of them are well known : continuity (resp. δ-continuity) $\Rightarrow$ almost continuity S & S $\Rightarrow$ θ-continuity $\Rightarrow$ weak-continuity $\Rightarrow$ rare-continuity. These are indeed weaker on independent forms of continuity. For example any function from any topological space to any hyperconnected space[20], i. e. the space in which all nonempty open subsets are dense is δ-continuous.

We define in this paper a new weak form of continuity under the title of strongly rare-continuity which is weaker than weak-continuity and stronger than rare-continuity. We determine some of its characteristics and some of its relations with the others. Some certain properties of strongly rarely continuous functions in terms of α-topologies of Njastad are also established.

PREPARATIONS

No specific separation axiom is assumed unless otherwise is explicitly stated. $\mathcal{N}_x$ will denote the family of whole basic neighbourhoods of the point x in the space X. cl

$A = \bar{A}$, int $A = \mathring{A}$ and ∂A denote respectively the closure, the interior and the boundary of the subset A in X. A function $f : X \to Y$ is called weakly continuous (resp. θ-continuous, resp. δ-continuous, resp. almost continuous $H.$, resp. almost continuous S and S, resp. rarely continuous) at $x \in X$ iff for each $W_{f(x)} \in \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ such that $f(G_x) \subseteq \overline{W}_{f(x)}$ (resp. $f(\bar{G}_x) \subseteq \overline{W}_{f(x)}$, resp. f (int $\bar{G}_x) \subseteq$ int $\overline{W}_{f(x)}$, resp. $G_x \subseteq \mathrm{cl} f^{-1}(W_{f(x)})$, resp. $f(G_x) \subseteq$ int $\overline{W}_{f(x)}$, resp. int $f(G_x) \subseteq \overline{W}_{f(x)}$) holds. The alpha and beta operations are defined as $\alpha\ A =$ int $\bar{A}$ and $\beta\ A =$ cl $\mathring{A}$ after Bourbaki[4]. As usual a subset A will be called nowhere dense iff $\alpha\ A = \phi$. We write $x \in A^*$ for a subset $A \subseteq X$ iff each $G_x \in \mathcal{N}_x$ is intersecting A with a non nowhere dense intersection, i. e. int $\overline{G_x \cap A} \neq \phi$. Then A^* is always closed, $\overline{A^*} = A^* \subseteq \bar{A}$ and $(A \cup B)^* = A^* \cup B^*$. Notice also that $\alpha A \subseteq A^*$ and therefore $\overline{\alpha A} \subseteq A^*$ hold since if a $U_x \in \mathcal{N}_x$ satisfy $U_x \subseteq \bar{A}$ then $\overline{G_x \cap A} \supseteq \mathrm{cl}\,(G_x \cap \overline{U_x \cap A}) \supseteq G_x \cap U_x$ could be written for any $G_x \in \mathcal{N}_x$. Since $\alpha\,(G \cap A) \overset{1}{=} \alpha(G \cap \alpha A)$ holds for any subset A and for any open G, one easily obtain $A^* \subseteq \overline{\alpha A}$. Thus $A^* = \overline{\alpha A}$ is found for any A. Notice that A is nowhere dense iff $A^* = \phi$ iff A^* is nowhere dense. The equality $(G \cap A^*)^* \overset{2}{=} (G \cap A)^{**}$ could also be proved for any open G and for any A. Hence the folfowing equivalences hold for any open G and for any A : $G \cap \alpha A = \phi$ iff $G \cap A^* = \phi$ iff $G \cap A^*$ is nowhere dense iff $G \cap A$ is nowhere dense. The equalities (1) and (2) and the last equivalencies are also valid for any semi-open subset G.

$k_* A = A \cup A^*$ is therefore a closure operation on X by a theorem of Andrijevic[1] and even $k_* A = \mathrm{cl}_\alpha A$ holds[1] for each $A \subseteq X$. Here $\mathrm{cl}_\alpha A$ denotes the closure of A in the τ_α topology of Njastad[15], i. e. the topolgy on X which is precisely the family of those sets written as the difference of an open set and a nowhere dense set of X.

Since we use the τ_α topology in some frequency throughout the paper, we give here a short and independent proof that the family τ_α is indeed a topology on X. Notice that if G_μ is open and N_μ is nowhere dense in X for each index μ then

$$G_\mu - N_\mu \subseteq \alpha\, G_\mu = \text{int}\, \beta\, (G_\mu - N_\mu) \subseteq \text{int}\, \beta\, (\bigcup_\mu\, (G_\mu - N_\mu))$$

and thus

$$\bigcup_\mu\, (G_\mu - N_\mu) = \text{int}\, \beta\, (\mu\, (G_\mu - N_\mu)) - N_0$$

is found where the nowhere dense N_0 is defined as

$$N_0 = \text{int}\, \beta\, (\bigcup_\mu\, (G_\mu - N_\mu)) - \bigcup_\mu\, (G_\mu - N_\mu).$$

Njastad proved that $\tau \subseteq \tau_\alpha$ and the nowhere dense subsets of these topologies are the same and therefore note that $A^*(\tau) = A^*(\tau_\alpha)$ for each $A \subseteq X$[15]. The space on the se t X equipped with the τ_α topology wlll briefly be denoted in this paper by X_α and a subset is called α-closed iff it is closed in X_α. The substar set of exclusively an inverse set under any function $f: X \to Y$ is defined as follows : $x \in (f^{-1}(B))^*$ iff $(f(G_x) \cap B)^* = \phi$ for each $G_x \in \mathcal{N}_x$. Then all substar sets are closed,

$$(f^{-1}(B))^* \cup (f^{-1}(B))_* \subseteq \text{cl} f^{-1}(B)$$

$$(f^{-1}(B \cup C))_* = (f^{-1}(B))_* \cup (f^{-1}(C))_*$$

$$(f^{-1}(B))_* = \phi \text{ if } B^* = \phi$$

and so $(f^{-1}(\bar{U}))_* = (f^{-1}(U))_*$ for each open $U \subseteq Y$. If the domain of f is compact then the equivalency $(f^{-1}(\bar{U}))_* = \phi$ iff $(U \cap f(X))_* = \phi$ holds for each open $U \subseteq Y$. The locally thinly scattered points of a function $f: X \to Y$ will be written by N_f and defined as follows : $x \in N_f$ iff $\exists\ G_x \in \mathcal{N}_x$; $(f(G_x))^* = \phi$. It is obvious that N_f is always open (may be empty) and satisfy $N_f \cap (f^{-1}(B))_* = \phi$ for all $B \subseteq Y$ E^1 and E^1_u will denote the one dimensional Euclidean space and the one dimensional upper limit topology of Sorgenfrey on reals respectively. We define simple and step functions just as in Royden[18]. In particularly the function $f(x) = [x] = \sup((-\infty, x] \cap Z)$ for each real x will be called as the classical step function where Z denotes the set of all integers. Semi-open (resp. regularly closed) subsets are those sets satisfying $A \subseteq \beta A$ (resp. $A = \beta A$) or equivalently those sets satisfying $G \subseteq A \subseteq \bar{G}$ (resp. $A = \bar{G}$) where G is an open set. H-closed spaces are those T_2 spaces which all their open coverings admit a finite and dense subfamilies or equivalently all their embeddings into Hausdorff spaces are closed[21]. Almost compact spaces are those which all their open coverings admit a finite subfamily such that the closures of members is a covering. Therefore H-closed spaces are precisely the almost compact T_2 spaces.

Strongly Rarely Continuous Functions

Theorem 1—The following are equivalent for any $f: X \to Y$.

(1) For each $W_{f(x)} \in \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ with $\alpha f(G_x) \subseteq \bar{W}_{f(x)}$ or equivalently $\alpha f(G_x) \subseteq \alpha W_{f(x)}$.

(2) For each $W_{f(x)} \in \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ such that $f(G_x) - W_{f(x)}$ or equivalently $f(G_x) - \bar{W}_{f(x)}$ is nowhere dense.

(3) For each $W_{f(x)} \in \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ and a nowhere dense $N_W \subseteq Y$ with $f(G_x) \subseteq \bar{W}_{f(x)} \cup N_W$ or equivalently $f(G_x) \subseteq W_{f(x)} \cup N_W$.

(4) For each $W_{f(x)} \subseteq \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ with $(f(G_x))^* \subseteq \bar{W}_{f(x)}$.

Proof : (1) $\Rightarrow$ (2) Easy since $\alpha(f(G_x) - W_{f(x)}) \subseteq \alpha f(G_x) - \bar{W}_{f(x)}$. Furthermore one can always write that

$$f(G_x) - W_{f(x)} = (f(G_x) - \bar{W}_{f(x)}) \cup (f(G_x) \cap \partial W_{f(x)})$$

where the second set participating the union is always nowhere dense.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are straightforward implications.

(3) ⇒ (4) Easy by taking the stars of the both sides of the inclusion relation of the hypothesis of (3).

Definition 1—A function f satisfying one of the conditions of Theorem 1 is called strongly rarely continuous at $x \in X$. Global definition of strongly rare-continuity could be given easily and expectedly. From now on we shortly write SRC for this type of functions.

Remark 1 : A function f is SRC (resp. rarely continuous) at $x \in X$ iff for each given $W_{f(x)} \in \mathcal{N}_{f(x)}$ there exists a $G_x \in \mathcal{N}_x$ such that the part of the image of G_x scattering to the outside of $W_{f(x)}$ is nowhere dense (resp. rare, i. e. has an empty interior). Hence any function with the nowhere dense (resp. rare) image such as simple and step function is evidently SRC (resp. rarely continuous). Not all the SRC functions have nowhere dense images. Here follows an example : Let Q be the set of all rationals and P be the set of all irrationals and let the space X on reals be defined so that all irrational singletons are open and the basic neighbourhoods of any $x \in Q$ are $U_x(\epsilon) =]x - \epsilon, x + \epsilon[\cap Q$. Then the function f defined as $f(x) = [x]\chi_Q(x) + x. \chi_P(x)$ from X into E^1 is SRC. It is not weakly continuous on integers and $f(X)$ is not nowhere dense in E^1. Here χ_A denotes the characteristic function of A.

Remark 2 : One of the recent weaker form of weak-continuity has defined by Rose[17] as follows : $f : X \rightarrow Y$ is called subweakly continuous iff there is an open basis $\mathcal{B}$ of the topology on Y such that cl $f^{-1}(B) \subseteq f^{-1}(\bar{B})$ for each $B \in \mathcal{B}$. Rose proved that[17] holding of the inclusions cl $f^{-1}(U) \subseteq f^{-1}(\bar{U})$ for each open $U \subseteq Y$ is the necessary and sufficient condition for the weak-continuity of f. Subweak-continuity, weak-continuity and θ-continuity of an almost continuous H. Function are equivalent by the Theorem 6 and Theorem 10 of Rose[17]. Notice that the identic function from the cofinite topology on an infinite set X onto the discrete topology on X is subweakly continuous but not SRC. The classical step function from E^1 to E^1 is SRC but not subweakly continuous hence not weakly continuous. Thus subweak-continuity (resp. weak-continuity) and strongly rare continuity are independent of each other. Notice that the function $g : E^1 \rightarrow E^1_u$ defined as $g(x) = x. \chi_\theta(x)$ is rarely continuous but not s. rarely continuous where Q denotes the set of all rationals in above. Therefore the evident implications

weak-continuity ⇒ strongly rare-continuity ⇒ rare-continuity are not reversible. Strongly rare-continuity is also different with an another recent and weaker form of weak-continuity defined by Long and Herrington[11] under the title of faint-continuity. The function defined in Example 2 of Long and Herrington[11] is faintly continuous but not SRC and the classical step function is not faintly continuous since it is easy to prove that $\tau = \tau_\theta^{11}$ iff τ is a regular topology, i. e. continuity and faint-continuity are coincided for any function defined into a regular space.

Theorem 2—Let the function $f: X \to Y$ be defined. Then

(i) f is strongly rarely continuous at $x \in X$ iff $x \notin (f^{-1}(Y - \overline{W}_{f(x)}))_*$ for each $W_{f(x)} \in \mathcal{N}_{f(x)}$.

(ii) The set of all points of which f is not strongly rarely continuous is
$\cup\, [(f^{-1}(U))_* - f^{-1}(\bar{U}) : U \subseteq Y \text{ open}]$.

(iii) f is s. rarely continuous iff $(f^{-1}(U))_* \subseteq f^{-1}(\bar{U})$ for all open (or semi-open) $U \subseteq Y$.

(iv) f is strongly continuous iff $f: X \to Y_\alpha$ is Strongly rarely continuous.

(v) f is strongly rarely continuous iff all the restrictions $f \mid G_\mu$ (resp. $f \mid K_\mu$) are so where $(G_\mu)_\mu$ (resp. $(K_\mu)_\mu$) is an open (resp. locally finite closed) covering of X.

(vi) f is strongly rarely continuous iff f is so on an appropriate basic neighbourhoods of each point.

Proof : (i) Let f be strongly rarely continuous at $x \in X$ and $W_{f(x)} \in \mathcal{N}_{f(x)}$ be given. Then there exists a $G_x \in \mathcal{N}_x$ with $(f(G_x))^* - \overline{W}_{f(x)} = \phi$ and so $(f(G_x) \cap (Y - \overline{W}_{f(x)}))^* = \phi$ i. e. $x \notin (f^{-1}(Y - \overline{W}_{f(x)}))^*$ is found. Sufficiency can be proved just reversely.

(ii) If f is not strongly rarely continuous at $x \in X$ then one gets

$$x \in (f^{-1}(Y - \overline{W}_{f(x)}))_* - f^{-1}(\text{cl}\,(Y - \overline{W}_{f(x)})).$$

If conversely $x \in (f^{-1}(U))_* - f^{-1}(\bar{U})$ for an open $U \subseteq Y$ then there exists a $W_{f(x)} \in \mathcal{N}_{f(x)}$ with $\overline{W}_{f(x)} \cap U = \phi$ and so the supposition of Strongly rare-continuity at $x \in X$ yields the existence of a $G_x \in \mathcal{N}_x$ with $(f(G_x) \cap U)^* = \phi$ a contradictory result with $x \in (f^{-1}(U))_*$.

(iii) Let f be strongly continuous and $U \subseteq Y$ be semi-open. Then

$$(f^{-1}(U))_* = (f^{-1}(W))_* \cup (f^{-1}(U - W))_* \subseteq f^{-1}(\overline{W}) = f^{-1}(\bar{U})$$

where the open W satisfy $W \subseteq U \subseteq \overline{W}$ and so $U - W \subseteq \partial W$ holds. Sufficiency is clear after the previous item (ii),

(iv) Notice firstly that if U is open and $N^* = \phi$ in the space Y then $\text{cl}_\alpha\,(U - N) = \bar{U}$ holds by noticing $(U - N)^* = \overline{\alpha U - \beta N} = \bar{U}$. Hence the statement follows from the condition (2) of Theorem 1 by the equivalency of being nowhere dense in the spaces Y and Y_α.

(v) After the Theorem (3i) only the sufficiency of the second statement will be proved. Let all the restrictions $f \mid K_\mu$ be Strongly rarely continuous and $W_{f(x)} \in \mathcal{N}_{f(x)}$ be given. Then there exist the indexes $\mu_1, \mu_2, \ldots \mu_n$ and a $G_x \in \mathcal{N}_x$ such that

$$x \in \bigcap_{k \leq n} K_{\mu k},\ G_x \cap \bigcup_{\substack{\mu \neq \mu_k \\ k \leq n}} = \phi.$$

There also exist nowhere dense subsets $N_W^k \subseteq Y$ and neighbourhoods $V_x^k \in \mathfrak{N}_x$ with

$$f\,(V_x^k \cap K_{\mu k}) \subseteq \overline{W}_{f(x)} \cup N_W^k .$$

Then by introducing a $V_x \in \mathfrak{N}_x$ with $V_x \subseteq G_x \cap \bigcap_{k \leq n} V_x^k$ and the nowhere dense $N_W = \cup\ [N_W^k : k \leqslant n] \subseteq Y$ one easily gets $f\,(V_x) \subseteq W_{f(x)} \cup N_W$.

(vi) Follows from (*v*).

Theorem 4—(i) All restrictions and graph function of a strongly rarely continuous function are again strongly rarely continuous.

(ii) If $f: X \to Y$ is continuous and $g: Y \to Z$ is strongly rarely continuous then $g.f: X \to Z$ is strongly rarely continuous.

PROOF : (i) Notice that the graph function g_f of f satisfy $g_f = (i_X \times f) \circ j_X$ where $j_X(x) = (x, x)$. Thus if f is s. rarely continuous then g_f will also be strongly rarely continuous by Theorem 7i and Theorem 3ii.

(ii) For any $W_{gf(x)} \in \mathfrak{N}_{gf(x)}$, one could find a nowhere dense $N_W \subseteq Z$ and an $U_{f(x)} \in \mathfrak{N}_{f(x)}$ with $g\,(U_{f(x)}) \subseteq W_{gf(x)} \cup N_W$ by the Strongly rare-continuity of g. Since there also exists a $G_x \in \mathfrak{N}_x$ with $f\,(G_x) \subseteq U_{f(x)}$, the statement is now clear.

Theorem 4—(i) δ-continuity, almost continuity S & S, θ-continuity, weak-continuity, strongly rare-continuity and rare-continuity of an open function are equivalent.

(ii) A function f is continuous iff f is strongly rarely continuous and inverses of all closed nowhere dense subsets under f are closed.

(iii) A function f is weakly continuous iff f is s. rarely continuous and it is weakly continuous on $\bar{N}_f$.

PROOF : (i) Every open rarely continuous function is weakly continuous[12] and therefore is almost continuous S & S[19] and consequently is δ-continuous[13].

(ii) Necessity is clear and sufficiency follows from the following lemma and (2) of Theorem 1.

Lemma—The following are equivalent for any function f:

(1) f is continuous at $x \in X$.

(2) $\forall W_{f(x)} \in \mathfrak{N}_{f(x)}, \exists\ G_x \in \mathfrak{N}_x, f^{-1}\,(\text{cl}\,(f\,(G_x) - W_{f(x)}))$ is closed.

(3) $\forall W_{f(x)} \in \mathfrak{N}_{f(x)}, \exists\ G_x \in \mathfrak{N}_x, x \notin \text{cl}\,(G_x - f^{-1}\,(W_{f(x)}))$.

PROOF : Notice that (2) $\Rightarrow$ (3) by

$$G_x - f^{-1}(W_{f(x)}) \subseteq f^{-1}(\mathrm{cl}\,(f(G_x) - W_{f(x)})) \subseteq f^{-1}(Y - W_{f(x)}).$$

The other implications are straightforward.

(iii) Necessity is obvious. Now let $f : X \to Y$ be strongly rarely continuous. Then the inclusion $f^{-1}(U) \subseteq N_f \cup (f^{-1}(U))_*$ holds for each open $U \subseteq Y$. In fact if a point $x \in f^{-1}(U) - N_f$ satisfy $x \notin (f^{-1}(U))_*$ then there exist a $G_x \in \mathcal{N}_x$ and $W_{f(x)} \in \mathcal{N}_{f(x)}$ such that

$$(f(G_x))^* \subseteq \overline{W}_{f(x)} \subseteq U \cup \partial U, \quad (f(G_x))^* \cap U = \phi,$$

one therefore would get $(f(G_x))^* \subseteq \partial U$ and so $(f(G_x))^* = \phi$ which is contradicting with $x \notin N_f$. Hence if f is strongly rarely continuous then $X - WC_f \subseteq \bar{N}_f$ is obtained by Theorem 2 (iii) after noticing the formula

$$X - WC_f = \cup\,[\mathrm{cl}\,(f^{-1}(U)) - f^{-1}(\bar{U}) : U \subseteq Y \text{ open}]$$

which is true for any function f where WC_f denotes briefly the set of all weak-continuity points of f. Therefore if a strongly rarely continuous function satisfy $\bar{N}_f \subseteq WC_f$ then $X = WC_f$ follows.

Definition 2—The graph of a function $f : X \to Y$ is called strongly α-closed iff for each $(x, y) \notin G(f)$ there exists a couple $(G_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y$ so that $(f(G_x) \cap {}_y)U^* = \phi$ or equivalently $\alpha(f(G_x)) \cap U_y = \phi$ where as usually $G(f)$ denotes the graph of f.

Remark 3 : Functions with closed graph have strongly α-closed graph and functions with strongly α-closed graph have α-closed graph. In fact if $G(f) \subseteq X \times Y$ is closed then $(x, y) \notin G(f)$ implies the existence of a $(G_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y$ with $(G_x \times U_y) \cap G(f) = \phi$ or equivalently $f(G_x) \cap U_y = \phi$. The second statement follows easily from the basic inclusion on graphs $(A \times B) \cap G(f) \subseteq A \times (f(A) \cap B)$. In fact if $(x, y) \notin G(f)$ and f has strongly α-closed graph then there exists a $(G_x, U_y) \in \mathcal{N}_x \times \mathcal{N}_y$ with $((G_x \times U_y) \cap G(f))^* = \phi$ and so $(x, y) \notin (G(f))^* \cup G(f) = \mathrm{cl}_\alpha\, G(f)$ and therefore $\mathrm{cl}_\alpha\, G(f) = G(f)$ are obtained. The function mentioned in Remark 5 in the sequel has non closed but strongly α-closed graph. The function $f : E^1 \to E^1$ defined as $f(x) = x - \sup((-\infty, x] \cap Z)$ has α-closed but not strongly α-closed graph since $G(f) \subseteq E^1 \times E^1$ satisfy $(G(f))^* = \phi$ but $f(G_k) \cap U_1)^* \neq \phi$ holds for each $(k, 1) \notin G(f)$, $k \in Z$ where Z denotes as usually the set of all integers.

Remark 4 : It is known that a function with the closed graph defined into a H-closed space is not necessarily weakly continuous[6] but is rarely continuous.[12]

Theorem 5—(i) If $f : X \to Y$ has strongly α-closed graph then $(f^{-1}(K))_* \subseteq f^{-1}(K)$ holds for any almost compact $K \subseteq Y$.

(ii) Functions with strongly α-closed graph defined into almost compact spaces are strongly rarely continuous.

(iii) Functions with closed graph defined into H-closed spaces are strongly rarely continuous.

(iv) The following are equivalent for functions defined into H-closed spaces:

(1) f is strongly rarely continuous.

(2) The graph of f is strongly α-closed.

(3) $(f^{-1}(K))_* \subseteq f^{-1}(K)$ hold for all almost compact $K \subseteq Y$.

PROOF: (i) Let $x \notin f^{-1}(K)$. Then there exists a couple $(G_x(y), U_y) \in \mathfrak{N}_x \times \mathfrak{N}_y$ with $(f(G_x(y)) \cap U_y)^* = \phi$ and therefore $(f(G_x(y)) \cap \bar{U}_y)^* = \phi$ for each $y \in K$. Since K is an almost compact subspace, there exists a $G_x \in \mathfrak{N}_x$ with $(f(G_x) \cap K)^* = \phi$. Hence $x \notin (f^{-1}(K))_*$ is established.

(ii) Take any open $U \subseteq Y$. Then $(f^{-1}(U))_* = (f^{-1}(\bar{U}))_* \subseteq f^{-1}(\bar{U})$ since all regularly closed subsets in an almost compact space are almost compact subspaces. The item now follows from Theorem 2(iii).

(iii) Direct consequence of (ii).

(iv) The implication (1) $\Rightarrow$ (2) will be proved in Theorem 6i. The others were already established.

Corollary (Long and Herrington[12])—Let $f: X \to Y$ be a function with closed graph where Y is H-closed. Then f is rarely continuous.

Theorem 6—(i) The graph of a strongly rarely continuous function defined into a T_2 space is strongly α-closed but not necessarily closed.

(ii) Images of compact subspaces under a function with strongly α-closed graph are α-closed.

(iii) Images of compact subspaces under a strongly rarely continuous function defined into a T_2 spaces are α-closed but not necessarily closed.

PROOF: (i) If $(x, y) \notin G(f)$ then there exists a couple $(W_y, G_x) \in \mathfrak{N}_y \times \mathfrak{N}_x$ with $\alpha f(G_x) \cap \bar{W}_y = \phi$ since f is strongly rarely continuous and Y is a T_2 space. Hence $\alpha(f(G_x) \cap W_y) = \phi$ and consequently $(f(G_x) \cap W_y)^* = \phi$ are found.

(ii) Let $K \subseteq X$ be compact and $y \notin f(K)$. Then there exists a $(G_x, U_y(x)) \in \mathfrak{N}_x \times \mathfrak{N}_y$ with $(f(G_x) \cap U_y(x))^* = \phi$ or equivalently $(f(G_x))^* \cap U_y(x) = \phi$ for each $x \in K$ since $G(f)$ is strongly α-closed. Then it is easy to see that there exists a $U_y \in \mathfrak{N}_y$ with $(f(K))^* \cap U_y = \phi$ i. e. $(f(K) \cap U_y)^* = \phi$. Hence $y \notin (f(K))^*$ and so $y \notin \mathrm{cl}_\alpha\ f(K)$ arc found.

(iii) Follows from the first two items.

Remark 5: The classical step function proves that a strongly rarely continuous function does not necessarily have closed graph even it is defined between two metrizable spaces since $(k+1, k) \in \overline{G(f)} - G(f)$ holds for each integer k in this example. Notice also that the function f defined in Remark 1 from E^1 into the cofinite topology on reals is strongly rarely continuous (even δ-continuous) but the images of compact subspaces are not necessarily closed.

Remark 6 : One way for obtaining the new strongly rarely continuous functions from the old ones is to product them by the first item of the following theorem just as for continuous and weakly type of continuous functions another way is to take their compositions with continuous functions. Another way is to take their compositions with continuous functions just as in Theorem (3ii).

Theorem 7—(i) A product function is s. rarely continuous iff each factor function is so.

(ii) A function f defined into a product space is strongly rarely continuous if each composition of f with the projection mappings is strongly rarely continuous.

PROOF : (i) Let each $f_\nu : X_\nu \to Y_\nu$ be strongly rarely continuous and the point $x = (x_\nu)_\nu \in \Pi X_\nu$ is taken. Then the closure of any basic neighbourhood of $(y_\nu)_\nu = (\Pi f_\nu)(x) = (f_\nu(x_\nu))$ contains $\cap [\pi^{-1}_{\nu_k}(\overline{W}(y_{\nu_k}) : k \leqslant n]$ i.e. the closure of a base member of the product space where $W(y_{\nu_k}) \in \mathcal{N}(y_{\nu_k})$ in the space Y_{ν_k} and π_ν denotes the projection mapping onto X_ν. Therefore there exists a $G(x_{\nu_k}) \in \mathcal{N}(x_{\nu_k})$ with $\alpha f_{\nu_k}(G(x_{\nu_k})) \subseteq \overline{W}(y_\nu)$ for each $k \leqslant n$. Then one gets

$$\alpha(f_0(\bigcap_{k\leqslant n} \pi^{-1}_{\nu_k}(G(x_{\nu_k}))) \subseteq \alpha(\bigcap_{k\leqslant n} \pi^{-1}_{\nu_k}(f_{\nu_k}(G(x_{\nu_k})))$$

$$= \bigcap_{k\leqslant n} \pi^{-1}_{\nu_k}(\alpha f_{\nu_k}(G(x_{\nu_k}))$$

where f_0 denotes briefiy the product function πf_ν. Hence the product function f_0 is strongly rarely continuous at $x = (x_\nu)_\nu$. Now conversely let f_0 be strongly rarely continuous. Take any index γ and any point $x_\gamma \in X_\gamma$. Then one easily creates a point x in the product space with $\pi_\gamma(x) = x_\gamma$. So for any $W(f_\gamma(x_\gamma)) \in \mathcal{N}(f_\gamma(x_\gamma))$ one gets

$$\pi^{-1}_\gamma(\overline{W}(f_\gamma(x_\gamma))) \supseteq \alpha(f_0(\bigcap_{k\leqslant n} \pi^{-1}_{\nu_k}(G(x_{\nu_k})))$$

$$= \bigcap_{k\leqslant n} \pi^{-1}_{\nu_k}(\alpha f_{\gamma_k}(G(x_{\gamma_k}))) \cap \prod_\nu \alpha f_\gamma(X_\nu)$$

by the appropriately chosen neighbourhoods $G(x_{\nu_k}) \in \mathcal{N}(x_{\nu_k})$. Now whether the index γ satisfy $\gamma = \nu_k$ for a $k \leqslant n$ or not, the function f_γ will be strongly rarely continuous at $x_\gamma \in X_\gamma$ by taking the π_γ projections of the both sides.

(ii) Left to the reader.

Remark 7 : The first item of the following theorem is a slight generalization of Theorem 6 of Rose[17]. The second item, stated independently by Takashi Noiri and the author gives a different version of a result of Noiri which is expressed in one of our correspondencies : Rarely continuous semi-open functions[2] are weakly continuous. It is also a strength version of Theorem 4i.

Theorem 8—(i) $f: X \to Y$ is almost continuous H. iff $f(\bar{U}) \subseteq \text{cl}\, f(U)$ for all semi-open $U \subseteq X$.

(ii) Almost continuity S & S, weak-continuity and strongly rare-continuity of an almost open Rose function are equivalent.

Proof : (i) We give here an independent proof. Let $f: X \to Y$ be almost continuous H. and $U \subseteq X$ be semi-open. Take any $x \in \bar{U} = \beta U$. Then $W_{f(x)} \cap f(U) = f(U \cap f^{-1}(W_{f(x)}))$ is nonempty since

$$\text{cl}\,(U \cap f^{-1}(W_{f(x)})) \supseteq \text{cl}\,(\overset{\circ}{U} \cap \text{cl}\, f^{-1}(W_{f(x)})) \supseteq \beta U \cap \alpha(f^{-1}(W_{f(x)})) \neq \phi$$

are all nonempty for each $W_{f(x)} \in \mathscr{W}_{f(x)}$. Sufficiency is a consequence of Theorem 6 of Rose[17].

(ii) A function $f: X \to Y$ is called almost open Rose [17] iff $f(G) \subseteq \alpha f(G)$ holds for each open $G \subseteq X$, i. e. images of all open sets are preopen. So almost continuity S and S of almost open Rose, s. rarely continuous functions follows easily by Theorem ii.

Weak*continuity

We define the weak*-continuous function as the strongly rarely continuous function f with the nowhere dense or equivalently empty N_f set. Then it is easy to see that by Theorem 4 (iii) every weak*-continuous function is weakly continuous but not conversely. Notice that a strongly rarely continuous function is not necessarily weak*-continuous. This is immediate after considering the step functions. Therefore the following implications are not reversible

weak*-continuity ⇒ weak-continuity ⇒ strongly rare-continututy.

If f is weak*-continuous then $(f^{-1}(U))^* \subseteq (f^{-1}(U))_*$ holds for each semi-open $U \subseteq Y$. Hence by remembering the equivalency of $G \cap A_* \neq \phi$ iff $(G \cap A^* \neq \phi$ proved in preparations, we obtain the following characterization :

Lemma—$f: X \to Y$ is weak*-continuous iff $(G \cap f^{-1}(U))^* \neq \phi$ imply $(f(G) \cap U)^* \neq \phi$ for each open G and for each semi-open $U \subseteq Y$.

Hence the first item of the following theorem which is version of Theorem 8 (i) is derived.

Theorem 9—(i) A weak*-continuous $f: X \to Y$ is almost continuous H. iff $f(\bar{U}) \subseteq (f(U))^*$ for each semi-open $U \subseteq X$.

(ii) Injective weak*-continuous functions into T_2 spaces can only be defined on T_2 spaces.

(iii) Strongly rare-continuity and weak*-continuity are equivalent for almost open Rose functions.

(iv) Strongly rare-continuity and weak*-continuity are equivalent for almost open S & S almost continuous H. (resp. semi-continuous) functions.

PROOF : (i) Notice only that if almost continuous H. function f is weak*-continuous then the following will be obtained by using the same notation of the proof of Theorem 8 (i)

$$(\mathrm{cl}\,(U \cap f^{-1}\,(W_{f(x)})))^* \supseteq (\mathrm{cl}\,(\overset{\circ}{U} \cap \alpha f^{-1}\,(W_{f(x)})))^*$$
$$\supseteq \mathrm{cl}\,(\beta\, U \cap \alpha f^{-1}\,(W_{f(x)})).$$

(ii) Let $f : X \to X$ be injective and weak*-continuous and let Y be a T_2 space. Then if $x \neq \xi$ there exists a couple $(G_x, W_{f(\xi)}) \in \mathcal{N}_x \times \mathcal{N}_{f(\xi)}$ with $\alpha f\,(G_x) \subset \overline{W}_{f(\xi)}$ $\cap = \phi$ and so there exists a $U_\xi \in \mathcal{N}_\xi$ with $\alpha f\,(G_x) \cap \alpha f\,(U_\xi) = \phi$. Hence $\alpha f\,(G_x$ $U_\xi) = \phi$ and consequently $G_x \cap U_\xi = \phi$ are found by the hypothesis of $N_f = \phi$. So different points have disjoint basic neighbourhoods in X.

(iii) Notice only that $N_f = \phi$ holds for any almost open Rose function f since $f\,(G_x)$ is contained in $\alpha f\,(G_x)$ for any $G_x \in \mathcal{N}_x$.

(iv) These statements are straightforward consequences of Theorem 4.2 and Theorem 4.3 of Noiri's paper[14] after (iii).

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