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QUASI-STATIC RESPONSE OF A LAYERED HALF-SPACE TO SURFACE LOADS

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The closed form expressions for the stresses caused by a two-dimensional shear line load acting on the boundary of a semi-infinite medium consisting of a homogeneous elastic layer lying over a homogeneous elastic half-space are first derived. The correspondence principle of viscoelasticity is then used to obtain the quasi-static response when the elastic half-space is replaced by a Maxwell viscoelastic half-space. Numerical calculations performed indicate that the quasi-static stresses differ significantly from the corresponding static stresses when the medium is purely elastic.

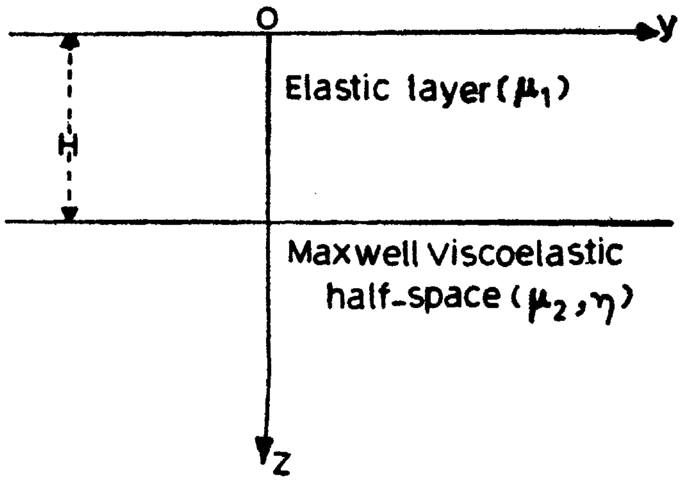
INTRODUCTION

The well known Boussinesq solution to the problem of a normal static load on the surface of a semi-infinite elastic medium offers wide applications to loading problems in geophysics and engineering. Transient crustal movement due to surface loading is of considerable interest in understanding the rheology of the earth's crust and upper mantle. This phenomenon is considered to be controlled by a quasi-static process of stress relaxation is viscoelastic regions within the earth. In quasi-static processes, the stress equilibrium exists at every point at each instant of time. This permits the neglect of the inertia term in the equation of motion. The quasi-static behaviour of the system is thus determined by the equation of equilibrium and equations relating stress, strain and displacement subject to boundary or initial conditions. The quasi-static deformation of a viscoelastic half-space by surface loads has been discussed by Lee[1], Fung[2]. Christensen[3], and others.

In the present paper, we obtain the static stresses due to a shear line load acting at the boundary of a semi-infinite medium which consists of a homogeneous, isotropic, elastic layer lying over a homogeneous, isotropic, elastic half-space. The correspondence principle of linear viscoelasticity is then used to obtain the quasi-static stresses when the half-space is Maxwell viscoelastic. In this model, the elastic layer represents the lithosphere of the earth and the Maxwell viscoelastic half-space represents the asthenosphere. Graphs for the shear stresses are drawn. It is found that these graphs differ significantly from the corresponding graphs for the elastic case.

2. Formulation of the Problem

We consider a model consisting of a homogeneous, isotropic, elastic layer of thickness H lying over a homogeneous, isotropic, Maxwell viscoelastic half-space. We place the origin of a cartesian coordinate system (x, y, z) at the boundary of the semi-infinite medium and the z-axis is drawn into the medium (Fig. 1). Let a shear line



Fig, 1. Section of the model by the plane $x = 0$.

load R per unit length be applied at the origin to the surface $z = 0$ in the positive direction of the x-axis. We shall be considering an antiplane strain problem in which the displacement components are given by

$$u = u(y, z), \; v = w \equiv 0.$$

We first calculate the shear stresses p_{12} and p_{13} at any point of the medium caused by a shear line load R acting on the surface $z = 0$ of the corresponding elastic model. We then use the corresponding correspondence principle of linear viscoelasticity to obtain the quasi-static response.

3. Elastostatic Solution

In the antiplane strain problem, the displacement n (for zero body force) satisfies the equation

$$\nabla^2 u = 0 \qquad \ldots(3.1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \qquad \ldots(3.2)$$

A suitable solution of (3.1) is of the type

$$u = \int_0^\infty \left(Ae^{-kz} + Be^{kz}\right)\left(\begin{array}{c}\sin ky \\ \cos ky\end{array}\right) dk \qquad \text{...(3.3)}$$

where A, B may be functions of k. Then

$$p_{12} = \mu \frac{\partial u}{\partial y} = \mu \int_0^\infty (Ae^{-kz} + Be^{kz})\left(\begin{array}{c}\cos k y \\ -\sin k y\end{array}\right) k\, dk \qquad \text{...(3.4)}$$

$$p_{13} = \mu \frac{\partial u}{\partial z} = \mu \int_0^\infty (-Ae^{-kz} + Be^{kz})\left(\begin{array}{c}\sin ky \\ \cos ky\end{array}\right) k\, dk \qquad \text{...(3.5)}$$

where μ is the rigidity of the medium.

We consider a semi-infinite medium consisting of a homogeneous, isotropic, elastic layer of thickness H lying over a homogeneous, isotropic, elastic half-space. It is assumed that the layer and the half-space are in welded contact. Let a shear line load R per unit length be applied at the origin to the surface $z = 0$ in the positive direction of the x-axis. Then the boundary condition at $z = 0$ is

$$p_{13} = -R\,\delta(y) \qquad \text{...(3.6)}$$

where $\delta(y)$ denote the Dirac delta fudction. We use the representation

$$\delta(y) = \int_0^\infty \cos k y\, dk. \qquad \text{...(3.7)}$$

Equations (3.6) and (3.7) suggest that we must choose the lower solution ($\cos ky$) in the expression (3.3) for u and other results related to u. The displacement u and the shear stress p_{13} at any point are

$$u(z) = \begin{cases} \int_0^\infty (A_1e^{-kz} + B_1e^{kz}) \cos ky\, dk & 0 \leqslant z \leqslant H \\ \int_0^\infty A_2e^{-kz} \cos ky\, dk & z \geqslant H \end{cases} \qquad \text{...(3.8)}$$

$$p_{13}(z) = \begin{cases} \mu_1 \int_0^\infty (-A_1e^{-kz} + B_1 e^{kz}) \cos ky\, k\, dk & 0 \leqslant z \leqslant H \\ -\mu_2 \int_0^\infty A_2e^{-kz} \cos ky\, k\, dk & z \geqslant H \end{cases} \qquad \text{...(3.9)}$$

where A_1, B_1 and A_2 are to be determined with the help of boundary conditions. μ_1 is the rigidity of the isotropic elastic layer and μ_2 is the rigidity of the isotropic elastic half-space. In the solution for the region $z \geqslant H$, the coefficient of $\exp(kz)$ is taken as zero, since, otherwise, $u \to \infty$ as $z \to \infty$. Using the boundary condition (3.6) and the continuity of the displacement (u) and the shear stress (p_{13}) across $z = H$, we can determine the coefficients A_1, B_1 and A_2. The substitution of the values of these coefficients in (3.8) gives the displacement u at any point of the medium. The corresponding stresses can then be obtained by simple differentiation. Expanding the denominator in a power series and evaluating the integrals, we find

$$p_{12} = \frac{-1}{\pi}\left[N_0\left(\frac{y}{y^2+z^2}\right) + \sum_{n=1}^{\infty} N_n\left\{\frac{y}{y^2+(2nH-z)^2} + \frac{y}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.10)$$

$$p_{13} = \frac{-1}{\pi}\left[N_0\left(\frac{z}{y^2+z^2}\right) - \sum_{n=1}^{\infty} N_n\left\{\frac{2nH-z}{y^2+(2nH-z)^2} - \frac{2nH+z}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.11)$$

for $0 \leqslant z < H$ and, for $z \geqslant H$

$$p_{12} = \frac{-2}{\pi}\left[M_0\left(\frac{y}{y^2+z^2}\right) + \sum_{n=1}^{\infty} M_n\left\{\frac{y}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.12)$$

$$p_{13} = \frac{-2}{\pi}\left[M_0\left(\frac{z}{y^2+z^2}\right) + \sum_{n=1}^{\infty} M_n\left\{\frac{2nH+z}{y^2+(2nH+z)^2}\right\}\right] \quad \ldots(3.13)$$

where

$$N_0 = R,\ M_0 = R\left(\frac{\mu^2}{\mu_1+\mu_2}\right) \quad \ldots(3.14)$$

$$N_n = R\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^n,\ M_n = R\left[\frac{\mu_2(\mu_1-\mu_2)^n}{(\mu_1+\mu_2)^{n+1}}\right]. \quad (3.15)$$

4. Viscoelastic Solution

We now use the correspondence principle[4] to obtain the quasi-static shear stresses for a model consisting of an elastic layer lying over a Maxwell viscoelastic half-space. For the elastic layer

$$p_{12} = 2\mu_1 e_{12},\ p_{13} = 2\mu_1 e_{13}. \quad \ldots(4.1)$$

For the Maxwell viscoelastic half-space

$$\dot{e}_{12} = \frac{1}{2\mu_2}\dot{p}_{12} + \frac{1}{\eta}p_{12} \qquad \ldots(4.2a)$$

$$\dot{e}_{13} = \frac{1}{2\mu_2}\dot{p}_{13} + \frac{1}{\eta}p_{13} \qquad \ldots(4.2b)$$

where η is viscosity and the dot ($\cdot$) signifies time-differentiation. Taking the Laplace transform of (4.2a, b) we obtain

$$s\,\bar{e}_{12} = \frac{s}{2\mu_2}\bar{p}_{12} + \frac{1}{\eta}\bar{p}_{12} \qquad \ldots(4.3a)$$

$$s\,\bar{e}_{13} = \frac{s}{2\mu_2}\bar{p}_{13} + \frac{1}{\eta}\bar{p}_{13} \qquad \ldots(4.3b)$$

where s is the Laplace transform variable. We may write (4.3a,b) in the form

$$\bar{p}_{12} = 2\mu_2^*\,\bar{e}_{12},\ \bar{p}_{13} = 2\mu_2^*\,\bar{e}_{13} \qquad \ldots(4.4)$$

where

$$\mu_2^* = \frac{s\mu_2}{s + 2\,\tau^{-1}} \qquad \ldots(4.5)$$

is the transform rigidity and $\tau = \eta/\mu_2$ is the relaxation time. Time-dependence of the load function is taken to be a step-function i. e.,

$$R\,(t) = R_0\,H\,(t) \qquad \ldots(4.6)$$

where $H\,(t)$ is the Heaviside step function. Then

$$\overline{R\,(t)} = \frac{R_0}{s}. \qquad \ldots(4.7)$$

In order to find the Laplace transformed solution of the viscoelastic problem, it is only necessary to replace μ_2 and R by μ_2^* and $\bar{R}$, respectively, in the corresponding elastic solution. From (3.10) – (3.13), we notice that μ_2 and R occur only in the expressions for N_0, N_n, M_0 and M_n. Therefore, the Laplace transformed solution of the viscoelastic problem is obtained from (3.10)—(3.13) on replacing N_0, N_n, M_0 and M_n by $\bar{N}_0$, $\bar{N}_n$, $\bar{M}_0$ and $\bar{M}_n$, respectively, where from (3.14), (3.15), (4.5) and (4.7).

$$\bar{N}_0 = \frac{R_0}{s},\ \bar{M}_0 = \frac{R_0}{C}\left(\frac{1}{s + A}\right) \qquad \ldots(4.8)$$

$$\bar{N}_n = R_0\,S_n\,(s),\ \bar{M}_n = \frac{R_0}{C}\,G_n\,(s) \qquad \ldots(4.9)$$

$$S_n\,(s) = \frac{(s\,B + A)^n}{s\,(s + A)^n},\ G_n\,(s) = \frac{(sB + A)^n}{(s + A)^{n+1}} \qquad \ldots(4.10)$$

$$A = \frac{2\mu_1}{(\mu_1 + \mu^2)\,\tau},\ B = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2},\ C = \frac{\mu_1 + \mu_2}{\mu^2}. \qquad ..(4.11)$$

In order to find the inverse Laplace transforms of $S_n(s)$ and $G_n(s)$, we use transform integrals listed in Erdélyi[5]. We find

$$L^{-1}[S_n(s)] = 1 + \exp(-At)\sum_{m=1}^{n} \frac{F_{2m}(-A)}{(n-m)!\,(m-1)!}\, t^{n-m} \qquad ...(4.12)$$

$$L^{-4}[G_n(s)] = \left[\frac{(1-B)^n (At)^n}{n!} + \sum_{m=1}^{n} \frac{B^m (1-B)^{n\cdot m}}{(n-m)!\,m!}(A\,t)^{n-m}\right]$$

$$\exp(-At) \qquad ...(4.13)$$

$$L^{-1}\left[\frac{1}{+A}\right] = \exp(-At) \qquad ..(4.14)$$

where

$$F_{2m}(s) = \frac{d^{m-1}}{ds^{m\ 1}}\left[\frac{(sB + A)^n}{s}\right]. \qquad ...(4.15)$$

Equations (3.10) – (3.13), (4.8), (4.9), (2.12) – (4.14) yield ($t > 0$)

$$p_{12} = \frac{-R_0}{\pi}\left[\frac{y}{y^2 + z^2} + \sum_{n=1}^{\infty}\left\{1 + \exp(-At)\sum_{m=1}^{n}\right.\right.$$

$$\left.\frac{F_{2m}(-A)}{(n-m)!\,(m-1)!}\, t^{n-m}\right\} \times \left\{\frac{y}{y^2 + (2nH + z)^2}\right.$$

$$\left.\left. + \frac{y}{y^2 + (2nH + z)^2}\right\}\right] \qquad ...(4.16)$$

$$p_{13} = \frac{-R_0}{\pi}\left[\frac{z}{y^2 + z^2} - \sum_{n=1}^{\infty}\left\{1 + \exp(-At)\sum_{m=1}^{n}\right.\right.$$

$$\left.\times \frac{F_{2m}(-A)}{(n-m)!\,(m-1)!}\, t^{n-m}\right\} \times \left\{\frac{2n\,H - z}{y^2 + (2nH - z)^2}\right.$$

$$\left.\left. - \frac{2n\,H + z}{y^2 + (2nH + z)^2}\right\}\right] \qquad ...(4.17)$$

for $0 < z < H$ and, for $z \geqslant H$

$$p_{12} = \frac{-2R_0}{\pi C}\left[\frac{y}{y^2 + z^2} + \sum_{n=1}^{\infty}\left\{\frac{y}{y^2 + (2nH + z)^2}\right\}\left\{\frac{(1-B)^n (A\,t)^n}{n!}\right.\right.$$

(equation continued on p. 627)

$$+ \sum_{m=1}^{n} \frac{B^m (1 - B)^{n-m}}{(n - m)!\, m!} (At)^{n-m} \Big\}\Big] \exp(- At) \qquad ...(4.18)$$

$$p_{13} = \frac{-2R_0}{\pi C}\Big[\frac{z}{y^2 + z^2} + \sum_{n=1}^{\infty}\Big\{\frac{2nH + z}{y^2 + (2nH + z)^2}\Big\}$$

$$\times \Big\{\frac{(1 - B)^n}{n!}(At)^n + \sum_{m=1}^{n}\frac{B^m (1 - B)^{n-m}}{(n - m)!\, m!}(At)^{n-m}\Big\}\Big]$$

$$\times \exp(At). \qquad ...(4.19)$$

Equation (4.16) and (4.17) give the quasi-static shear stresses at any point of an elastic layer of thickness H lying over a Maxwell viscoelastic half-space caused by a shear line load acting at the origin to the surface $z = 0$ in the positive direction of the x-axis. Equations (4.18) and (4.19) give the quasi-static shear stresses at any point of the Maxwell viscoelastic half-space.

5. Particular Case

We consider the particular case when the rigidities μ_1 and μ_2 are equal, i. e.,

$$\mu_1 = \mu_2 = \mu \text{ (say)}. \qquad ...(5.1)$$

Equations (4.11) and (5.1) give

$$A = \tau^{-1},\ B = 0,\ C = 2. \qquad ...(5.2)$$

Using (5.2) equation (4.15) yields

$$F_{2m}(-A) = \frac{-(m - 1)!}{\tau^{n-m}}. \qquad ...(5.3)$$

From (4.16) – (4.19), (5.2) and (5.3), we find ($t > 0$)

$$p_{12} = \frac{-R_0}{\pi}\Big[\frac{y}{y^2 + z^2} + \sum_{n=1}^{\infty}\Big\{1 - \exp(-t/\tau)\sum_{k=1}^{n-1}\frac{(t/\tau)^k}{k!}\Big\}$$

$$\times\Big\{\frac{y}{y^2 + (2nH - z)^2} + \frac{y}{y^2 + (2nH + z)^2}\Big\}\Big] \qquad ...(5.4)$$

$$p_{13} = \frac{-R_0}{\pi}\Big[\frac{z}{y^2 + z^2} - \sum_{n=0}^{\infty}\Big\{1 - \exp(-t/\tau)\sum_{k=0}^{n-1}\frac{(t/\tau)^k}{k!}\Big\}$$

$$\times\Big\{\frac{2nH - z}{y^2 + (2nH - z)^2} - \frac{2nH + z}{y^2 + (2nH + z)^2}\Big\}\Big] \qquad ...(5.5)$$

for $0 \leqslant z < H$ and

$$p_{12} = \frac{-R_0}{\pi}\left[\frac{y}{y^2+z^2} + \sum_{n=1}^{\infty} \frac{(t/\tau)^n}{n!}\left\{\frac{y}{y^2+(2n\,H+z)^2}\right\}\right]$$

$$\exp(-t/\tau) \qquad \text{...(5.6)}$$

$$p_{13} = \frac{-R_0}{\pi}\left[\frac{z}{y^2+z^2} \sum_{n=1}^{\infty} \frac{(t/\tau)^n}{n!}\left\{\frac{2nH+z}{y^2+(2n\,H+z)^2}\right\}\right]$$

$$\exp(-t/\tau) \qquad \text{...(5.7)}$$

for $z \geqslant H$.

The case when $t = 0$ corresponds to the elastic problem. Equations (3.10)—(3.13) and (5.1) then yield

$$p_{12} = \frac{-R_0}{\pi}\left(\frac{y}{y^2+z^2}\right) \qquad \text{...(5.8a)}$$

$$p_{13} = \frac{-R_0}{\pi}\left(\frac{z}{y^2+z^2}\right) \qquad \text{...(5.8b)}$$

for every value of z ($0 \leqslant z < \infty$) with $R = R_0$.

Since we have taken $\mu_1 = \mu_2 = \mu$, eqns. (5.8a, b), in fact, give the shear stresses at any point z within an elastic half-space due to a shear line load R per unit length acting at the origin to the surface $z = 0$ in the positive direction of the x-axis.

6. Numerical Results

In eqns. (5.4) and (5.5), we have obtained the expressions for the quasi-static shear stresses p_{12} and p_{13} within the medium ($0 \leqslant z \leqslant H$) caused by a shear line load acting at the origin to the surface $z = 0$ in the positive direction of the x-axis. We wish to study the variation of these stresses with horizontal distance y and time t for given values of z. For this purpose, we define the dimensionless quantities α, Y, T, P_{12} and P_{13} through the relations

$$z = \alpha\, H,\; y = YH,\; t = T\tau \qquad (6.1)$$

$$p_{12} = \frac{R_0}{\pi H} P_{12},\; p_{13} = \frac{R_0}{\pi H} P_{13}.$$

Using (6.1), equations (5.4) and (5.5) yield ($T > 0$)

$$p_{12} = -\frac{Y}{Y^2+\alpha^2} - \sum_{n=1}^{\infty}\left\{1 - \exp(-T)\sum_{k=0}^{n-1}\frac{T^k}{k!}\right\}\left\{\frac{Y}{Y^2+(2n-\alpha)^2}\right.$$

(*equation continued on p.* 629)

$$+ \frac{Y}{Y^2 + (2n + \alpha)^2} \Big\} \qquad \ldots(6.2)$$

$$p_{13} = - \frac{\alpha}{Y^2 + \alpha^2} + \sum_{n=1}^{\infty} \Big\{ 1 - \exp(-T) \sum_{k=0}^{n-1} \frac{T^k}{k!} \Big\} \Big\{ \frac{2n - \alpha}{Y^2 + (2n - \alpha)^2}$$

$$- \frac{2n + \alpha}{Y^2 + (2n + \alpha)^2} \Big\} \qquad \ldots(6.3)$$

where P_{12} and P_{13} are the dimensionless shear stresses. Y and T are, respectively, the dimensionless horizontal distance and the dimensionless time. For $T = 0$ eqns. (5.8a, b) give

$$P_{12} = - \frac{Y}{Y^2 + \alpha^2} \qquad \ldots(6.4)$$

$$P_{13} = - \frac{\alpha}{Y^2 + \alpha^2} \qquad \ldots(6.5)$$

as the dimensionless stresses for the elastic case. We note that

$$\frac{Y}{Y^2 + (2n - \alpha)} + \frac{Y}{Y^2 + (2n + \alpha)^2}$$

$$= \frac{2Y (4n^2 + Y^2 + \alpha^2)}{[Y^2 + (2n - \alpha)^2] [Y^2 + (2n + \alpha)^2]} \qquad \ldots(6.6)$$

and

$$\frac{2n - \alpha}{Y^2 + (2n - \alpha)^2} - \frac{2n + \alpha}{Y^2 + (2n + \alpha)^2}$$

$$= \frac{2\alpha [4n^2 - (Y^2 + \alpha^2)]}{[Y^2 + (2n - \alpha)^2] [Y^2 + (2n + \alpha)^2]} . \qquad \ldots(6.7)$$

Since $\Big\{ \exp(-T) \sum_{k=0}^{n-1} \frac{T^k}{k} \Big\} < 1$ for all values of n and $T > 0$, it is obvious that the infinite series appearing on the right hand sides of (6.2) and (6.3) converge at least as rapidly as the infinite series $\Sigma n^{\frac{1}{2}}$. In our numerical computation, we found that the first ten terms of this infinite series are adequate.

Figures 2-4 exhibit the variation of the dimensionless horizontal shear stress P_{12} with the dimensionless horizontal distance Y for three values of z, namely, $z = H$, $H/2$, 0 and three values of the dimensionless time T ($T = 0, 1, 10$). For all values of T and $z > 0$, $P_{12} = 0$ when $Y = 0$ and when $z = 0$, P_{12} tends to infinity as Y tends to zero [see eqns. (6.2) and (6.4)]. The graphs for $T = 0$ correspond to the elastic case. From (6.4), we find that, in the elastic case, the shear stress P_{12}, for all values

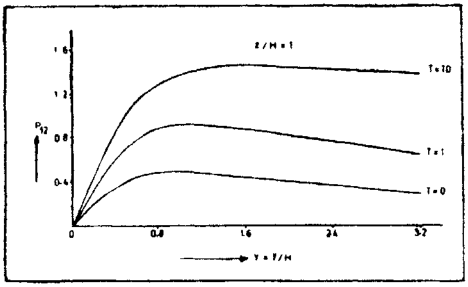


FIG. 2. Variation of the shear stress P_{12} with the horizontal distance Y when $z = H$.

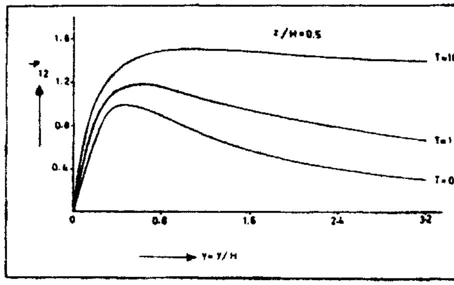


FIG. 3. Variation of the shear stress P_{12} with the horizontal distance Y when $z = H/2$.

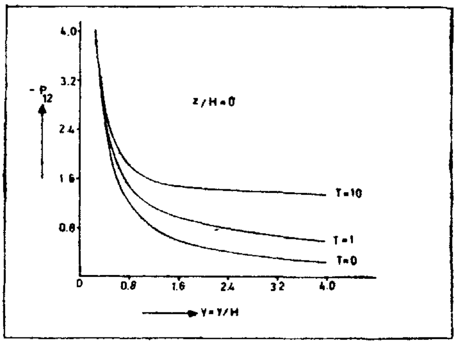


FIG. 4. Variation of the shear stress P_{12} with the horizontal distance Y when $z = 0$.

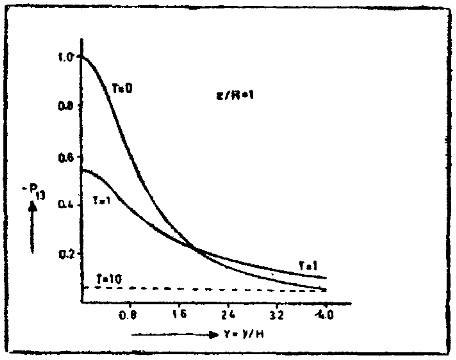


FIG. 5. Variation of the shear stress P_{13} with the horizontal distance Y when $z = H$.

of z, tends to zero as Y tends to infinity. We note that the deviation of the viscoelastic solution from the elastic solution increases as z increases for a given value of Y.

Figures 5-6 show the variation of the dimensionless shear stress P_{13} with the dimensionless horizontal distance Y for two values of z, namely, $z = H$, $H/2$ and different values of the dimensionless time T. The graphs for $T = 0$ correspond to the

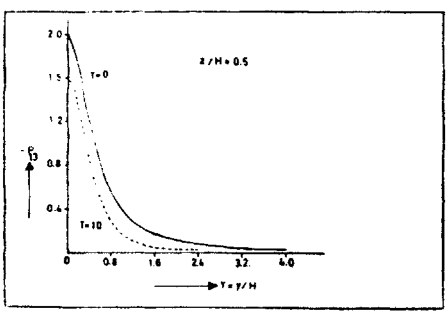


FIG. 6. Variation of the shear stress P_{13} with the horizontal distance Y when $z = H/2$.

elastic case for which $P_{13} = -H/z$ when $Y = 0$. Also, P_{13}, for all values of z, tends to zero as Y approaches infinity [see eqn. (6.5)]. For $z = H$, the graphs for various values of T are quite different from the graph for the elastic case (Fig. 5). For $z = H/2$, there is only a slight difference between the graphs for the viscoelastic case and the elastic case (Fig. 6).

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References

1. E. Lee, *Quart. J. Appl. Maths.* **13** (1955), 183-90.
2. Y. C. Fung, *Foundatians of Solid Machanics*. Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1965.
3. R. M. Christensen, *Theory of viscoelasticity : An Introduction*. Academic Press, New York, 1971.
4. W. Flüge, *Viscoelasticity*. Springer-Verlag, New York, 1975.
5. A. Erdélyi, *Tables of Integral Transforms. Volume I.* McGraw-Hill, New York, 1954.