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CIRCULAR ORBITS OF CHARGED TEST PARTICLES IN RIESSNER-NORDSTROM FIELD

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Criteria for the existence and stability of circular orbits of a charged particle in Reissner-Nordstrom field have been discussed using the path deviation equations. Vibration frequencies in the orbital plane and the plane perpendicular to it are determined and the Shirokov effect of the Einsteinian theory of gravitation is obtained in the presence of charges.

1. INTRODUCTION

In this paper we have discussed the existence and stability of circular orbits of a charged test particle in the Nordstrom field using the path deviation equations. In the absence of the central charge the discussion reduces to that of the motion of a particle in the Schwarzschild field and we get back the effect of the Einsteinian theory of gravitation obtained by Shirokov[1]. Even in the case of the motion of a neutral test particle in the field of the charged central body we get the additional gravitational effects due to the central charge obtained by Howes[2] for the Nordstrom metric in the absence of the cosmological constant.

A comparison of the two results, namely, the results of this paper and the one obtained by Shirokov reveals the additional graviational effects which are called into play due to the presence of the charges.

The equations of motion of charged test particle of mass m_0 and charge e_0 are given by generalized Lorentz equations

$$u^i_{;j}\, u^j = -\frac{e_0}{m_0} F^i_j \;\; u^j \qquad \ldots(1.1)$$

where u^i ($= dx^i/ds$) is the unit 4-velocity of the charged test particle and a semi-colon (;) denotes covariant differentiation. The path being given by eqns (1.1) the deviation equations with the help of the results given in Synge[3] are obtained in the form

$$\frac{d^2\xi^i}{ds^2} + 2\,\Gamma^i_{jk}\, u^j \left(\frac{d\xi^k}{ds}\right) + \frac{\partial}{\partial x^l}\left(\Gamma^i_{jh}\right) u^j u^k \xi^l$$

$$= -\frac{e_0}{m_0}\left\{F^i_j\left(\frac{d\xi^j}{ds}\right) + 2\,F^i_j\,\Gamma^j_{mk}\xi^m u^k + \frac{\partial}{\partial x^n}\left(F^i_j\right)\xi^n u^j\right\} \qquad ...(1.2)$$

where ξ^i is the small four vector whieh gives the deviation of the test particle from the fiducial path

2. Circular Orbits in the Nordstrom Field

The gravitational field of a central charged body is described by the Reissner-Nordstrom metric

$$ds^2 = -\left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right)^{-1} d\gamma^2 - \gamma^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$+\left(1 - \frac{2m}{\gamma^2} + \frac{\epsilon^2}{\gamma^2}\right) dt^2. \qquad ...(2.1)$$

The only surviving component of the eletromagnetic field tensor F_{ij} is $F_{41} = \epsilon/\gamma^2$. The arbitrary constants of integration m and ϵ appearing in eqn. (2.1) are identified as mass and charge of the gravitating body respectively.

In order to determine the expressions for basic velocities in the Nordstrom field we suppose that the fiducial path is a circular trajectory with γ = constant in the plane $\theta = \pi/2$. This gives $u^1 = d\gamma/ds = 0$ and $u^2 = d\theta/ds = 0$; and the only non-vanishing components of the velocity are $u^3 = d\phi/ds$ and $u^4 = dt/ds$. From equations of motion (1.1) and the line-element (2.1) we get respectively

$$(u^3)^2 = \left(\frac{m}{\gamma^3} - \frac{\epsilon^2}{\gamma^4}\right)^2 (u^4) + \frac{e_0}{m_0}\left(\frac{\epsilon}{\gamma^3}\right) u^4 \qquad ...(2.2)$$

and

$$(u^3)^2 = \frac{\left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right)(u^4)^2 - 1}{\gamma^2}. \qquad ..(2.3)$$

From these equations the explicit expressions for the components u^3 and u^4 are obtained as

$$u^3 = \frac{\Big[\left(\frac{2m}{\gamma} - \frac{2\epsilon^2}{\gamma^2}\right)\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right) + \left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right) \times \left\{\left(\frac{e_0}{m_0}\frac{\epsilon}{\gamma}\right)^2 \pm \frac{e_0}{m_0}\frac{\epsilon}{\gamma}\right.}{\sqrt{2}\,\gamma\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right)}$$

(*equation continued on p.* 643)

$$\times \left(4 \left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right) + \frac{e_0^2}{m_0^2} \frac{\epsilon^2}{\gamma^2} \right)^{1/2} \Big\}\Big] \qquad ...(2.4)$$

and

$$u^4 = \frac{\frac{e_0}{m_0} \frac{\epsilon}{\gamma} \pm \left(4\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right) + \frac{e_0^2}{m_0^2} \frac{\epsilon^2}{\gamma^2} \right)^{1/2}}{2\left(1 - \frac{3m}{\gamma} + \frac{2\epsilon^2}{\gamma^2}\right).} \qquad ...(2.5)$$

For the circular orbits to exist u^3 and n^4 must be real i. e.

$$(u^3)^2 > 0 \text{ and } (u^4)^2 > 0 \qquad ...(2.6)$$

and this determines the existence region. Which is same as obtained by Howes[2] in the absence of the cosmological constant for aneutral test particle. Therefore we may conclude that charge of the particle does not have any effeet on existence region.

To discuss the stability of the circular orbits we impart a small momentum distrubance $d\xi^i/ds$ to the orbiting test particle. If stable, its consequent vibrations may be in the orbital plane and also in the plane perpendicular to it. Since θ disturbances are independent of γ, ϕ, t perturbations we consider equations (1.2) with $i = 2$ in order to discuss the stability in the plane perpendicular to the orbital plane.
This gives

$$\frac{d^2 \xi^2}{ds^2} + (u^3)^2 \xi^2 = 0. \qquad ...(2.7)$$

The solution of this equation is

$$\xi^2 = \xi_0^2 \, e^{i\Omega s} \qquad ...(2.8)$$

where

$$\Omega^9 = (u^3)^2. \qquad ...(2.9)$$

In the region where the circular orbits, will be stable to disturbances perpendicular to orbital plane for real Ω which is the situation in view of eqns. (2.6) and (2.9). The periods of these vibuations will be $2\pi/\Omega$.

To examine stability in the orbital plane we consider eqns. (1.2) with $i = 1, 3, 4$ and obtain,

$$\frac{d^2 \xi^1}{ds^2} + a_1 \frac{d\xi^3}{ds} + a_2 \frac{d\xi^4}{ds} + a_3 \xi' = 0 \qquad ...(2.10)$$

$$\frac{d^2 \xi^3}{ds^2} + b \frac{d\xi^1}{ds} = 0 \qquad ...(2.11)$$

and

$$\frac{d^2 \xi^4}{ds^2} + C_1 \frac{d\xi^1}{ds} + C_2 \xi^3 + C_3 \xi^4 = 0 \qquad \ldots(2.12)$$

where

$$a_1 = -2\gamma \left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right) u^3 \qquad \ldots(2\ 13)$$

$$a_2 = \left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^2}\right) \left\{\left(\frac{2m}{\gamma^2} - \frac{2\epsilon^2}{\gamma^3}\right) u^4 + \frac{e_0}{m_0} \frac{\epsilon}{\gamma^2}\right\} \qquad \ldots(2.14)$$

$$a_3 = \left(-1 + \frac{\epsilon^2}{\gamma^2}\right)(u^3)^2 + \left\{\frac{6m^2}{\gamma^4} + \frac{5\epsilon^4}{\gamma^6} - \frac{12m\,\epsilon^2}{\gamma^5} - \frac{2m}{\gamma^3} + \frac{3\,\epsilon^2}{\gamma^4}\right\} \times (u^4)^2 + \left\{\frac{6m}{\gamma^4} - \frac{4\epsilon^3}{\gamma^5} - \frac{2\epsilon}{\gamma^3} + \frac{2e_0}{m_0} \times \left(\frac{m}{\gamma^2} - \frac{\epsilon^2}{\gamma^3}\right)\right\} u^4 \qquad \ldots(2.15)$$

$$b = \frac{2}{\gamma} u_8 \qquad \ldots(2.16)$$

$$c_1 = \frac{1}{\left(1 - \frac{2m}{\gamma} + \frac{\epsilon^2}{\gamma^3}\right)} \left\{\left(\frac{2m}{\gamma^2} - \frac{2\epsilon^2}{\gamma^3}\right) u^4 + \frac{c_0}{m_0} \frac{\epsilon}{\gamma^2}\right\} \qquad \ldots(2.17)$$

$$c_2 = -\frac{2e_0}{m_0} \frac{\epsilon}{\gamma} u_3 \qquad \ldots(2.18)$$

$$c_3 = \frac{2e_0}{m_0} \frac{\epsilon}{\gamma^2} \left(\frac{m}{\gamma^2} - \frac{\epsilon^2}{\gamma^3}\right) u^4. \qquad \ldots(2.19)$$

In order to solve eqns. (2.10) – (2.12) we make use of the substitutions

$$\xi^1 = \xi^1_0\, e^{i\omega s};\ \xi^3 = \xi^3_0\, e^{i\omega s} \text{ and } \xi^4 - \xi^4_0\, e^{i\omega s} \qquad \ldots(2.20)$$

is eqns. (2.10) - (2.12) and obtain

$$(a_3 - \omega^2)\, \xi^1_0 + a_1 (i\,\omega)\, \xi^3_0 + a_2 (i\omega)\, \xi^4_0 = 0 \qquad \ldots(2.21)$$

$$b\ (i\omega)\ \xi^1_0 + (-\omega^2)\, \xi^3_0 = 0 \qquad \ldots(2.22)$$

and

$$c_1\ (i\omega)\ \xi^1_0 + c_2\, \xi^3_0 + (c_3 - \omega^2)\, \xi^4_0 = 0. \qquad \ldots(2.23)$$

For these equations to yield non-trivial solutions we require

$$\begin{vmatrix} (a_3 - \omega^2) & a_1 i\omega & a_2 i\omega \\ bi\omega & -\omega^2 & 0 \\ c_1 i\omega & c_2 & (c_3 - \omega^2) \end{vmatrix} = 0 \qquad \ldots(2.24)$$

This gives on simplification the square of the vibration frequency in the orbital plane as $\omega^2 = 0$ and

$$\omega^2 = \frac{-(a_2 c_1 + a_1 b - a_2 - c_3) \pm [(a_2 c_1 + a_1 b - a_3 - c_3)^2 - 4(a_3 c_3 + a_2 bc_2 - a_1 bc_3)]^{1/2}}{2} \qquad \ldots(2.25)$$

where a_1, a_2, a_3, b, c_1, c_2 and c_3 are given by equations (2.13) – (2.19). Stability to disturbances in the orbital plane requires $\omega^2 > 0$ and the period of these vibrations is given by $2\pi/\omega$. The Shirokov general relativistic effect in the presence of charges is given by the difference in the periods of vibrations namely, by $2\pi/\Omega - 2\pi/\omega$ where Ω and ω are given by eqns. (2.9) and (2.25) respectively. Although the existence region of the circular orbits and the stabilty to the disturbances prependicular to the orbital plane are independent of the terms involving e_0/m_0 but the stability to disturbances in the orbital plane is effected by terms involving e_0/m_0. However it is difficult to express the ettect of such terms explicitly. If we take $e_0 = 0$ in the above equations we get the results obtained by Howes[2] for a neutral test particle in the Nordstrom field with $\lambda = 0$ and if we put $\epsilon = 0$ in the above equations we get the results obtained by Shirokovo[1].

References

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2. J. Robert Howes, *GRG* **13** (1991), 829–35.
3. J. L. Synge, *Relativity : The General Theory*, North Holland Publishing Company, p. 19.