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RANDOM RAYLEIGH WAVES IN NON-HOMOGENEOUS ELASTIC MEDIA

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The problem of propagation of Rayleigh waves in a semi-infinite elastic medium having a vertical non-homogeneity has been considered following Beltzer[1]. The variances of the displacements and velocities for a stationary white noise at a point on the boundary has been numerically evaluated.

1. INTRODUCTION

Earthquakes and random material defects give rise to Rayleigh waves of strengths randomly varying with frequency. Beltzer[1] considered such waves for elastic and viscoelastic media. He assumed the medium to be homogeneous. It is natural therefore to ask how the response will be modified for a non-homogeneity of known type. We have investigated here the problem of random waves generated in elastic non-homogeneous medium. In order to make the problem tractable the vertical non-homogeneity is assumed to be of a type admitting decoupling of the equations of motion. The variance of the displacements and velocities are obtained and are compared with the values obtained by Beltzer[1].

2. RAYLEIGH WAVES IN NON-HOMOGENEOUS ISOTROPIC MEDIUM

Let us consider the propagation of a plane wave through an isotropic elastic non-homogeneous half-space with a free plane boundary.

For simplicity we take the boundary as $z = z_1$ with positive z towards the interior of the solid and take the plane wave travelling in the x-direction. Stoneley [see Ewing p. 350] took the equations for two dimensional motion as

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda\theta + 2\mu \frac{\partial u_x}{\partial x}\right) + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\right\} \quad ...(1)$$

$$\rho \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left(\lambda\theta + 2\mu \frac{\partial u_z}{\partial z}\right). \quad ...(2)$$

Taking $\lambda = \mu =$ a function of z only and ρ is also a function of z only,

$$\text{put } u_x = u'_x/\sqrt{\mu} \quad \text{and} \quad u_z = u'_z/\sqrt{\mu} \quad ...(3)$$

then the eqns. (1) and (2) become

$$\frac{\rho}{\sqrt{\mu}} \frac{\partial^2 u'_x}{\partial t^2} = 3\sqrt{\mu}\frac{\partial^2 u'_x}{\partial x^2} + 2\sqrt{\mu}\frac{\partial^2 u'_z}{\partial z\,\partial x} + \sqrt{\mu}\frac{\partial^2 u'_x}{\partial z^2} + \frac{u'_x}{4\mu^{3/2}}\left(\frac{d\mu}{dz}\right)^2 - \frac{u'_x}{2\sqrt{\mu}}\frac{d^2\mu}{dz^2} \quad \text{...(4)}$$

$$\frac{\rho}{\sqrt{\mu}} \frac{\partial^2 u'_z}{\partial t^2} = \sqrt{\mu}\frac{\partial^2 u'_z}{\partial x^2} + 2\sqrt{\mu}\frac{\partial^2 u'_x}{\partial z\,\partial x} + 3\sqrt{\mu}\frac{\partial^2 u'_z}{\partial z^2} + \frac{3u'_z}{4\mu^{3/2}}\left(\frac{d\mu}{dz}\right)^2 - \frac{3u'_z}{2\sqrt{\mu}}\frac{d^2\mu}{dz^2}. \quad \text{...(5)}$$

Taking $\mu = \mu_0 (z/z_0)^2$ and $\rho = \rho_0 (z/z_0)^2$ ...(6)

where μ_0, ρ_0 are constants, then (4) and (5) reduce to

$$\rho_0 \frac{\partial^2 u'_x}{\partial t^2} = \mu_0\left[3\frac{\partial^2 u'_x}{\partial x^2} + 2\frac{\partial^2 u'_z}{\partial z\,\partial x} + \frac{\partial^2 u'_x}{\partial z^2}\right] \quad \text{...(7)}$$

$$\rho_0 \frac{\partial^2 u'_z}{\partial t^2} = \mu_0\left[\frac{\partial^2 u'_z}{\partial x^2} + 2\frac{\partial^2 u'_x}{\partial z\,\partial x} + 3\frac{\partial^2 u'_z}{\partial z^2}\right]. \quad \text{...(8)}$$

Taking

$$u'_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}\,;\quad u'_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}. \quad \text{...(9)}$$

ϕ, ψ being functions of x, z and t only, then, from (7) and (8), we get

$$\alpha_0^2\,\nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} \quad \text{...(10)}$$

$$\beta_0^2\,\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2} \quad \text{...(11)}$$

where

$$\alpha_0^2 = \frac{3\mu_0}{\rho_0}\,;\quad \beta_0^2 = \frac{\mu_0}{\rho_0} \quad \text{...(12)}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

Assuming a wave travelling in the x-direction of wave length $2\pi/k$, We get

$$\phi = g(z) \exp(i k (x - ct))$$
$$\psi = h(z) \exp(i k (x - ct)). \qquad ...(13)$$

Substituting (13) in (10) and (11) we get

$$g(z) = A_1 e^{-rz} \text{ and } h(z) = A_2 e^{-sz}$$

where

$$\left. \begin{aligned} r^2 &= k^2\left(1 - \frac{c^2}{\alpha_0^2}\right); \; c < \alpha_0 \\ s^2 &= k^2\left(1 - \frac{c^2}{\beta_0^2}\right); \; c < \beta_0 \end{aligned} \right\} \qquad ...(14)$$

$$\therefore \quad \phi = A_1 e^{-rz} \exp(i k (x - ct)) \qquad ...(15)$$

$$\psi = A_2 e^{-rz} \exp(i k (x - ct)). \qquad ...(16)$$

The boundary conditions

$$\tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \quad \text{on } z = z_1, \text{ give}$$

$$\left. \begin{aligned} A_1\, i\, k\, e^{-rz_1} (2rz_1 + 1) + A_2 e^{-sz_1} [(k^2 + s^2) z_1 + s] = 0 \\ A_1 e^{-rz_1} [z_1 (3r^2 - k^2) + 3r] - A_2\, i\, k\, e^{-sz_1} (2z_1 s + 3) = 0 \end{aligned} \right\}. \qquad ...(17)$$

Eliminating A_1 and A_2 we get the frequency equation

$$z_1^2 \,|\, k \,|^2 \left[4\sqrt{\left(1 - \frac{c^2}{\beta_0^2}\right)\left(1 - \frac{c^2}{3\beta_0^2}\right)} - \left(2 - \frac{c^2}{\beta_0^2}\right)^2 \right]$$
$$+ z_1 \,|\, k \,|\, \frac{c^2}{\beta_0^2} \left(3\sqrt{1 - \frac{c^2}{3\beta_0^2}} + \sqrt{1 - \frac{c^2}{\beta_0^2}} \right)$$
$$+ 3 - 3\sqrt{\left(1 - \frac{c^2}{\beta_0^2}\right)\left(1 - \frac{c^2}{3\beta_0^2}\right)} = 0. \qquad ...(18)$$

The above equation is consistent with values of c/β_0 greater than a fixed number ξ (the fixed number ξ represents the Rayleigh wave velocity in a homogeneous medium with β_0 as S wave velocity). Hence the possible Rayleigh waves in the medium under consideration have velocities ranging from ξ to 1, and the corresponding frequency from ω_0 to ∞ (ω_0 is the frequency corresponding to wave velocity ξ). We shall consider random Rayleigh waves with frequencies lying in the above range.

Now we can write the displacement u_x and u_z in the following way

$$\left. \begin{aligned} u_x &= BF_x(x, z)\, e^{-ikct} \\ u_z &= BF_z(x, z)\, e^{-ikct} \end{aligned} \right\} \qquad ...(19)$$

where

$$\left.\begin{aligned} F_x &= i\,k\sqrt{\mu}\,\frac{z}{z_1}\left[e^{-rz} - \frac{s\,(2rz_1+1)}{z_1\,(k^2+s^2)+s}.e^{-rz_1}.e^{s(z_1-z)}\right]e^{ikx} \\ F_z &= \sqrt{\mu}\,\frac{z}{z_1}\left[-re^{-rz} + \frac{k^2\,(2rz_1+1)}{z_1\,(k^2+s^2)+s}.e^{-rz_1}.e^{s(z_1-z)}\right]e^{ikx} \end{aligned}\right\}. \quad \text{...(20)}$$

3. Basic Equations in the Random Case

Making use of the arbitrary nature of the value B we can introduce a random complex process $B(\omega)$, $\omega = kc$, with zero mean and uncorrelated increments such that for any interval (ω_1, ω_2) (Beltzer[1]).

$$< | B(\omega_2) - B(\omega_1) |^2 > = \int_{\omega_1}^{\omega_2} S_B(\omega)\,d\omega \quad \text{...(21)}$$

where < > denotes averaging and S_B is the spectral density. The stationary random Rayleigh waves are now defined as stochastic integrals which describe the superposition of the waves given by (19)

$$u_i(x, z, t) = \int_{-\infty}^{\infty} F_i(\omega, x, z)\,e^{-i\omega t}\,dB(\omega),\ (i = x, z) \quad \text{...(22)}$$

where F_i are given in (20).

The waves defined by (22) can be described also in the form of stochastic integrals

$$u_i(x, z, t) = \int_{-\infty}^{\infty} e^{-i\omega t}\,dE_i(\omega, x, t),\ (i = x, z) \quad \text{...(23)}$$

where $E_i(\omega, x, z)$ describes complex amplitudes at apoint (x, z) for random processes with zero mean and uncorrelated increments.

By comparison between (22), (23) we get

$$E_i(\omega, x, z) = \int_{-\infty}^{\omega} F_i(\delta, x, z)\,dB(\delta),\ (-\infty < \omega < \infty) \quad \text{.. (24)}$$

and the spectra s_E^i are given by

$$S_E^i(\omega, x, z) = S_B(\omega)\,|\,F_i(\omega, x, z)\,|^2,\ (i = x, z). \quad \text{...(25)}$$

We consider the random motion at the point $(0, z_1)$ to be prescribed. We get the complex random amplitudes and the spectra during propagation from (24), (25) as

$$E_i(\omega, x, z) = \int_{-\infty}^{\infty} f_i(\delta, x, z)\, dE_i(\delta, 0, z_1) \qquad \text{...(26)}$$

and

$$S_E^i(\omega, x, t) = S_E^i(\omega, 0, z_1)\, |f_i(\omega, x, z)|^2 \qquad \text{...(27)}$$

where $E_i(\omega, 0, z_1)$ and $S_E^i(\omega, 0, z_1)$ are the amplitudes and the spectra at $(0, z_1)$ and

$$f_i(\omega, x, z) = F_i(\omega, x, z)/F_i(\omega, 0, z_1), \; (i = x, z). \qquad \text{...(28)}$$

The pr ocesses $u_i(x, z, t)$ are not independent and (24), (25) yield the coupling equations between their random amplitudes and their spectra :

$$E_x(\omega, x, z) = \int_{-\infty}^{\infty} \psi_{xz}(\delta, z)\, dE_z(\delta, x, z) \qquad \text{...(29)}$$

and

$$S_E^x(\omega, x, z) = S_E^z(\omega: x, z)\, |\psi_{xz}(\omega, z)|^2 \qquad \text{...(30)}$$

where

$$\psi_{xz}(\omega, z) = F_x(\omega, x, z)/F_z(\omega, x, z). \qquad \text{...(31)}$$

4. Random Rayleigh Waves in Non-homogeneous Elastic Media

The functions $f_i(\omega, x, z)$ and $\psi_{xz}(\omega, x, z)$, which govern the evaluation of the processes $u_i(x, z, t)$ and the coupling between them can be written according to eqns. (20), (28), (31) in the following way :

$$f_x(\omega, x, z) = \frac{z}{z_1}\left[\frac{e^{r(z_1-z)} - Pse^{s(z_1-z)}}{1 - Ps}\right] e^{ikx} \qquad \text{...(32)}$$

$$f_z(\omega, x, z) = \frac{z}{z_1}\left[\frac{-re^{r(z_1-z)} + Pk^2 c^{s(z_1-z)}}{-r + Pk^2}\right] e^{ikx} \qquad \text{...(33)}$$

$$\psi_{xz}(\omega, z) = ik\left[\frac{e^{r(z_1-z)} - Pse^{s(z_1=z)}}{-re^{r(z_1-z)} + rk^2 e^{s(z_1-z)}}\right] \qquad \text{...(34)}$$

where

$$P = \frac{2rz_1 + 1}{z_1(k^2+s^2)+s} \qquad \text{...(35)}$$

and r, s are given in (14).

Let $\bar{S}$ be the intensity of a white noise disturbance in the x-direction at $r = (0, z_1)$:

$$S_E^x \ (\omega, 0, z_1) = \bar{S}; \ | \ \omega \ | < \infty. \qquad \ldots(36)$$

Then from (31) & (34) we get

$$S_E^z (\omega, 0, z_1) = \frac{\bar{S}(-r + Pk^2)}{k^2 (1 - Ps)^2} . \qquad \ldots(37)$$

Then we get, the variances of the displacements and their nth derivatives as

$$\text{Var} \ [u_i^{(n)} (x, z)] = \int_{-\infty}^{\infty} \omega^{2n} \ S_E^i \ (\omega, x, z) \ d\omega$$

$$= \int_{-\infty}^{\infty} \omega^{2n} \ | \ f_i \ (\omega, x, z) \ |^2 \ S_E^i \ (\omega, o, z_1) \ d\omega. \qquad \ldots(38)$$

So, we have the result :

$$\text{Var} \ [u_x^{(n)} (x, z)] = \int_{-\infty}^{\infty} \omega^{2n} \left[\frac{z}{z_1} \ \frac{e^{r(z_1 - z)} - Pse^{s(z_1 - z)}}{1 - Ps} \right]^2 \bar{S} \ d\omega \qquad \ldots(39)$$

and

$$\text{Var} \ [u_z^{(n)} (x, z)] = \int_{-\infty}^{\infty} \omega^{2n} \left[\frac{z}{z_1} \ \frac{-re^{r(z_1 - z)} + Pk^2 \ e^{s(z_1 - z)}}{k \ (1 - Ps)} \right]^2 \bar{S} \ d\omega \ldots(40)$$

where P, r, s, are given in (35), (14).

5. Discussion

The variance of u_x, u_z and $\frac{du_x}{dt}$, $\frac{du_z}{dt}$ have been plotted in Fig. 1 and Fig. 2 respectively against z/z_1. Curves of the form var $(u_x) =$ constant$/z^n$, and similarly for others, have been fitted to the data in order to compare them with the homogeneous case of Beltzer[1]. The values of n obtained are as follows :

n	u_x	u_z	$\frac{du_x}{dt}$	$\frac{du_z}{dt}$
Homogeneous case (actual value)	1.0	1.0	3.0	3.0
Non-homogeneous case (data based)	0.334	2.106	1.020	2.474

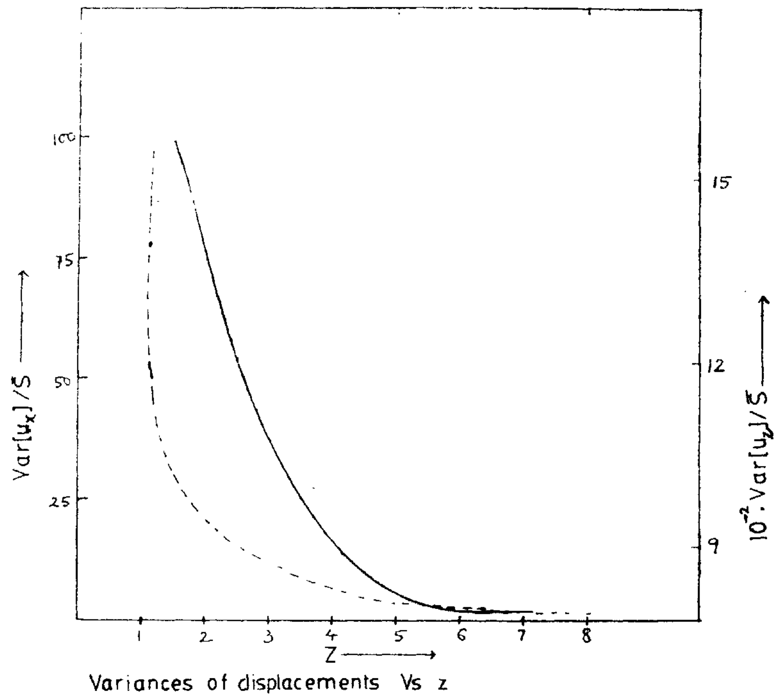

FIG. 1. Variances of displacements *Vs z*.

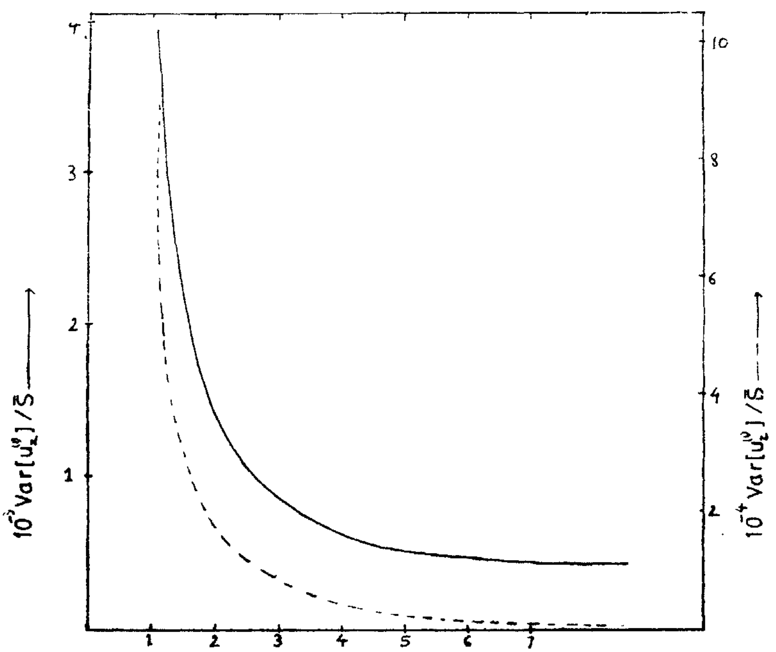

FIG. 2. Variances of velocities *Vs z*.

The vertical non-homogeneity considered in this problem therefore diminishes the change of variance of u_x, $\frac{du_x}{dt}$, $\frac{du_z}{dt}$ with depth compared to the homogeneous case, while var (u_z) increases with depth relative to the homogeneous case.

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