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ON THE REAL ROOTS OF A RANDOM ALGEBRAIC POLYNOMIAL

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Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of identically distributed independent random variables belonging to the domain of attraction of the symmetric stable law (Gnedenko and Kolmogorov, p. 171). Suppose $a_0, a_1, \cdots a_n$ are non-zero real numbers and $\max_{0 \leqslant r \leqslant n} |a_r| = k_n$, $\min_{0 \leqslant r \leqslant n} |a_r| = t_n$ and $\frac{k_n}{t_n} = O(\log n)$. If $N_n(\omega)$ be the number of real roots of the equation $\sum_{r=0}^{n} a_r X_r x^r = 0$, then for all $n > n_0$, it is proved that

$$P\left\{\omega: N_n(\omega) < \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}\right\}$$

$$< \frac{161\ e \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}{\log n}.$$

INTRODUCTION

§1. The object of this paper is to find the lower bound of $N_n(\omega)$ of the random polynomial

$$f(x) = \sum_{r=0}^{n} a_r X_r(\omega) x^r \qquad \text{...(1)}$$

when the coefficients are not identically distributed and belong to the domain of attraction of the symmetric stable law with index 'α' i. e. $0 < \alpha \leqslant 2$. The problem of finding bounds for $N_n(\omega)$ has been considered by various authors. Such type of equation has been considered by Dunnage[3]. Samal and Mishra[5] have considered the same case when the variance is infinite. Mishra, *et al.*[6] considered the lower bound for $N_n(\omega)$ when the coefficients X_r's belong to the domain of attraction of the normal law. In this paper we will extend the results of Mishra *et al.*[6] to the case of domain of attraction of the symmetric stable law.

Theorem—Let $f(x) = \sum_{r=0}^{n} a_r X_r x^r$ be a polynomial of degree n where X_r's are identically distributed independent random variables belonging to the domain of attraction of the symmetric stable law with characteristic function $\exp\{-|t|^{\alpha} h(t)\}$,

$0 < \alpha \leqslant 2$, $h(t)$ being a positive slowly varying function in the neighbourhood of the origin. Let $a_0, a_1, a_2, \ldots a_n$ be non-zero real numbers. Then there exists a positive integer n_0 such that for all $n > n_0$,

$$P\left\{\omega : N_n(w) < \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}\right\}$$

$$< \frac{161\, e \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}{\log n}$$

provided $\lim_{n\to\infty}\left(\frac{k_n}{t_n}\right)$ is finite, where $k_n = \max_{0<r\leqslant n} |a_r|$ and $t_n = \min_{0<r\leqslant n} |a_r|$.

§2. For the proof of the theorem we need the following definitions, notations and lemmas in the sequel.

Let M be the integer defined by

$$M = \left[\left(\frac{k_n}{t_n}\right)^{\alpha} (\log n)^5\right] + 1 \qquad \ldots(2)$$

where $[x]$ denotes the greatest integer $\leqslant x$; and let k be determined by

$$M^{2k} \leqslant n \leqslant M^{2k+2}. \qquad \ldots(3)$$

From (2) and (3) it follows that

$$\frac{\log n}{4 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)} \leqslant k \leqslant \frac{\log n}{2 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}. \qquad \ldots(4)$$

We consider $f(x) = \sum_{r=0}^{n} a_r X_r x^r$ at the points

$$x_m = \left(1 - \frac{1}{M^{2m}}\right)^{1/\alpha} \qquad \ldots(5)$$

for $m = \left[\frac{k}{2}\right] + 1, \left[\frac{k}{2}\right] + 2, \ldots k.$

There are $k/2$ points if k is even and $(k+1)/2$ points if k is odd. We express $f(x_m)$ for sufficiently large n as sum of the three parts as follows :

$$f(x_m) = \left(\sum_1 + \sum_2 + \sum_3\right) a_r X_r(\omega)\, x_m^r$$

where the index r ranges from $M^{2m-1} + 1$ to M^{2m+1} in $\sum_1$, from 0 to M^{2m-1} in $\sum_2$ and from $M^{2m+1} + 1$ to n in $\sum_3$.

We write

$$A_m(\omega) = \sum_1 a_r X_r(\omega) x_m^r \qquad \text{...(6)}$$

and

$$R_m(\omega) = (\sum_2 + \sum_3) \; a_r X_r(\omega) x_m^r \,. \qquad \text{...(7)}$$

Obviously A_m and A_{m+1} are independent random variables.

Lemma 2.1—If $h(t)$ is a positive slowly varying function in the neighbourhood of the origin, then for $\rho > 0$,

$$\text{(i)} \quad \lim_{t \to 0} \; t^{\rho} h(t) = 0$$

$$\text{(ii)} \quad \lim_{t \to 0} \; t^{-\rho} h(t) = \infty.$$

These results follow from Karamata's theorem (cf. Ibragimov and Linnik[2], p. 795). We define normalizing constants V_m by

$$V_m^{\alpha} = \sum_1 |a_r|^{\alpha} x_m^{\alpha r} h(a_r x_m^r \theta / V_m)$$

where θ is an arbitrary small positive number.

Lemma 2.2—$V_m > t_n \left(\frac{d}{e} M^{2m} \right)^{1/\alpha}$ for $d > 0$.

PROOF : $h(t)$ may be bounded or unbounded when $t \to 0$. If $\lim_{t \to 0} h(t) = \infty$, then there exists $t_1 > 0$ such that $h(t) > 1$, for $t < t_1$. If $h(t)$ is bounded, as $h(t)$ is positive in the neighbourhood of the origin, there exists $d > 0$ such that $h(t) > d$. Hence $h(t) > d$ for both the cases. So for large n, we get

$$d > \frac{1}{\log n \log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)^2} \cdot \qquad \text{...(8)}$$

Now

$$\begin{aligned} V_m^{\alpha} &= \sum_1 |a_r|^{\alpha} x_m^{\alpha r} h(a_r x_m^r \theta / V_m) \\ &> \sum_{M^{2m-1}+1}^{2m+1} d \, x_m^{\alpha r} \, t_n^{\alpha} \\ &> \sum_{M^{2m-1}+1}^{M^{2m}} d \, x_m^{\alpha r} \, t_n^{\alpha} \end{aligned}$$

(*equation continued on p.* 658)

$$= d\, t_n^{\alpha} \;(M^{2m} - M^{2m-1})\left(l - \frac{1}{M^{2m}} \right)^{M^{2m}}$$

$$> \frac{d}{e}\, t_n^{\alpha} \;M^{2m}.$$

Thus $V_m > t_n \left(\frac{d}{e} M^{2m} \right)^{1/\alpha}.$ ...(9)

Lemma 2.3—Let

$$T_1 = \left\{ \omega : \left| \sum_3 a_r X_r(\omega)\, x_m^r \right| > \frac{V_m}{2} \right\}$$

$$T_2 = \left\{ \omega : \left| \sum_2 a_r X_r(\omega)\, x_m^r \right| > \frac{V_m}{2} \right\}$$

$$T = \{ \omega : | R_m(\omega) | > V_m \}$$

and G be the set of all points for $m = \left[\frac{k}{2} \right] + 1, \left[\frac{k}{2} \right] + 2, \ldots k$ where $| R_m(\omega) | > V_m$.

Then

$$P(G) \leqslant 129\, e \;\frac{\log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)}{\log n}.$$

PROOF : The characteristic function of $\sum_3 a_r X_r(\omega)\; x_m^r$ is given by

$$\phi_m(t) = \exp \{ - | t |^{\alpha} h_m(t) \}$$

where

$$h_m(t) = \sum_3 | a_r |^{\alpha} x_m^{\alpha r} h \left(a_r x_m^r t \right).$$

Now

$$P(T_1) < 2 - \left| \frac{V_m}{4} \int_{-4/V_m}^{4/V_m} \phi_m(t)\, dt \right|$$

(cf. Gnedenko and Kolmogorov, p. 54)

$$< \frac{V_m}{4} \int_{-4/V_m}^{4/V_m} | 1 - \phi_m(t) |\, dt.$$

But

$$h_m(t) = \sum_3 | a_r |^{\alpha} x_m^{\alpha r} h(a_r\, x_m^r\, t)$$

(*equation continued on p.* 659)

$$< | t |^{-\rho} \sum_{3} | a_r |^{\alpha-\rho} x_m^{r(\alpha-\rho)} .$$

Since by Lemma 2.1, for $\rho > 0$,

$$h(a_r x_m^r t) \leqslant | a_r x_m^r t |^{-\rho} \text{ as } t \to 0.$$

But

$$\begin{aligned}(1 - \phi m(t)) &= 1 - \exp(- | t |^{\alpha} hm(t)) \\ &= | t |^{\alpha} hm(t)(1 + o(1)) \text{ as } t \to 0 \\ &< 2 | t |^{\alpha-\rho} \sum_{3} | a_r |^{\alpha-\rho} x_m^r (\alpha - \rho).\end{aligned}$$

Now

$$P(T_1) \leqslant V m \sum_{3} x_m^{r(\alpha-\rho)} | a_r |^{\alpha-\rho} \int_0^{4/V_m} t^{\alpha-\rho} dt$$

$$= \frac{2^{2\alpha-2\rho+2}}{\alpha-\rho+1} \frac{\sum_{3} x_m^{r(\alpha-\rho)} | a_r |^{\alpha-\rho}}{V_m^{\alpha-\rho}}$$

$$< \frac{2^{2\alpha-2\rho+2}}{\alpha-\rho+1} \frac{\sum_{3} x_m^{r(\alpha-\rho)} k_n^{\alpha-\rho}}{V_m^{\alpha-\rho}}.$$

But

$$\sum_{3} x_m^{r(\alpha-\rho)} \leqslant \frac{2\alpha}{\alpha-\rho} M^{2m-1} .$$

So

$$P(T_1) \leqslant 2^{2\alpha+3} \frac{\alpha}{(\alpha-\rho)(\alpha-\rho+1)} \left(\frac{e}{d}\right)^{1-\rho/\alpha}$$

$$\times \left(\frac{k_n}{t_n}\right)^{\alpha-\rho} \frac{1}{\left(\frac{k_n}{t_n}\right)^{\alpha\left(1-\frac{2m\rho}{\alpha}\right)} (\log n)^{5\left(1-\frac{2m\rho}{\alpha}\right)}}$$

(by Lemma 2.2 and (4))

$$< \frac{512}{\alpha+2} \left(\frac{e}{d}\right) \left(\frac{k_n}{t_n}\right)^{\alpha} \frac{1}{\left(\frac{k_n}{t_n}\right)^{(\alpha-2m\rho)} \log n^{\left(5-\frac{10m\rho}{\alpha}\right)}}$$

(*equation continued on p.* 660)

$(\because \frac{k_n}{t_n} > 1$ for large $n)$

$$\leqslant 171\left(\frac{e}{d}\right)\left(\frac{k_n}{t_n}\right)^{2m\rho}\frac{1}{(\log n)^{\left(\epsilon - \frac{10m\rho}{\alpha}\right)}}$$

$$\alpha \geqslant 1$$

$$\left(\because \frac{1}{\alpha+2} < \frac{1}{3}\right).$$

Choose $\rho = \frac{\alpha}{10m}$ for a fixed m and by this choice we have,

$$P(T_1) < 171\left(\frac{e}{d}\right)\frac{1}{(\log n)^3}. \quad \ldots(10)$$

Then proceeding as in above we get

$$P(T_2) \leqslant \frac{2^{2\alpha-2\rho+2}}{\alpha-\rho+1}\frac{\sum_2 x_m^{r(\alpha-\rho)} \mid a_r \mid^{\alpha-\rho}}{V_m^{\alpha-\rho}}.$$

But

$$\sum_2 x_m^{r(\alpha-\rho)} \leqslant 1 + M^{2m-1} \leqslant 2M^{2m-1}.$$

So

$$P(T_2) \leqslant 86\left(\frac{e}{d}\right)\frac{1}{(\log n)^3}. \quad \ldots(11)$$

Now

$$\mid R_m \mid = \left| \sum_2 a_r X_r(\omega) x_m^r + \sum_3 a^r X_r(\omega) x_m^r \right|$$

$$\leqslant \left|\sum_2 a_r X_r(\omega) x_m^r\right| + \left|\sum_3 a_r X_r(\omega) x_m^r\right|$$

$$\leqslant \frac{V_m}{2} + \frac{V_m}{2} = V_m.$$

Obviously,

$$P(T) \leqslant P(T_1) + P(T_2)$$

$$\leqslant 257\left(\frac{e}{d}\right)\frac{1}{(\log n)^3}$$

(by (10) and (11)).

Then

$$P(G) \leqslant \sum_{m=1}^{k} P(T) = k\, P(T)$$

$$\leqslant 257 \left(\frac{e}{d}\right) \frac{k}{(\log n)^3}$$

$$< 129\, e \frac{\log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)}{\log n} \qquad \ldots(12)$$

(by (4) and (8)).

§ 3. Proof of the Theorem—Let

$$E_m = \{\omega\colon A_{2m} \geqslant V_{2m},\ A_{2m+1} < -V_{2m+1}\}$$

$$F_m = \{\omega\colon A_{2m} < -V_{2m},\ A_{2m+1} \geqslant V_{2m+1}\}$$

$$D_1 = \{\omega\colon A_{2m} \geqslant V_{2m},\ |R_{2m}| < V_{2m}\}$$

$$D_2 = \{\omega\colon A_{2m+1} < -V_{2m+1},\ |R_{2m+1}| < V_{2m+1}\}$$

$$D_3 = \{\omega\colon A_{2m} < -V_{2m},\ |R_{2m}| < V_{2m}\}$$

$$D_4 = \{\omega\colon A_{2m+1} \geqslant V_{2m+1},\ |R_{2m+1}| < V_{2m+1}\}$$

and

$$H_1 = D_1 \cap D_2,\ H_2 = D_3 \cap D_4.$$

Obviously when H_1 occurs, $f(x_{2m}) > 0$ and $f(x_{2m+1}) < 0$, and also when H_2 occurs, $f(x_{2m}) < 0$ and $f(x_{2m+1}) > 0$.

Therefore, if $H_1 \cup H_2$ occurs, $f(x)$ has a root in (x_{2m}, x_{2m+1}). Again we are to show that $P(E_m \cup F_m) > 0$.

Let $G_m(x)$ and $g_m(t)$ be respectively the distribution function and the characteristic function of (A_m/V_m). Then

$$P(E_m \cup F_m) = (1 - G_m(1))\ G_{2m+1}(-1) + G_{2m}(1)\ (1 - G_{2m+1}(1)). \qquad \ldots(13)$$

Also

$$g_m(t) = \exp\left\{ -|t|^{\alpha} \frac{1}{V_m^{\alpha}} \sum_{1} a_r\, x_m^{\alpha r}\, h\left(x_m^{r}\, t/V_m\right)\right\}$$

$$= \exp\left\{ -|t|^{\alpha} \frac{1}{V_m^{\alpha}} \quad \frac{\theta}{t} \mid o(1) \sum_{1} a_r\, x_m^{\alpha r}\ h\left(x_m^{r}\ \theta/V_m\right)\right.$$

(*equation continued on p. 662*)

$$(1 + o(1)) \Big\} \text{ (by lemma (2.1))}$$

$$= \exp \{ | - t |^{\alpha - o(1)} \theta^{o(1)} (1 + o(1)) \}$$

(by the definition of V_m).

Therefore as $m \to \infty$, $g_m(t) \to \exp(- | t |^{\alpha})$ in any bounded interval of t-values.

Hence

$$\sup_x | G_m(x) - F(x) | = o(1). \quad ...(14)$$

So for $\epsilon > 0$, we have

$$| G_{2m}(-1) - F(-1) | < \epsilon$$

and

$$| G_{2m+1}(-1) - F(-1) | < \epsilon.$$

So

$$G_{2m}(-1) > F(-1) - \epsilon, \quad G_{2m+1}(-1) > F(-1) - \epsilon$$

and

$$1 - G_{2m}(1) > 1 - F(1) - \epsilon, \ 1 - G_{2m+1}(1) > 1 - F(1) - \epsilon.$$

Hence

$$P(E_m \cup F_m) \geqslant 2 (F(-1) - \epsilon)(1 - F(1) - \epsilon)$$

$$\to 2 F(-1)(1 - F(1) \text{ as } \epsilon \to 0 \text{ when } m \to \infty \text{ with } n.$$

Thus

$$P(E_m \cup F_m) = \delta_m = \delta > 0. \quad ...(15)$$

Where δ is an absolute constant.

Let η_m be the indicator function of the event $E_m \cup F_m$.

Then by (15)

$$P\{\omega: \eta_m = 1\} = \delta_m \text{ and } P\{\omega: \eta_m = 0\} = 1 - \delta_m.$$

Thus η_m's are independent random variables with

$$E(\eta_m) = \delta_m \text{ and } V(\eta_m) = (\delta_m - \delta_m^2).$$

It follows from Samal[4] (p. 439) that there are at least

$$\tfrac{1}{2} \left\{ k - \left[\frac{k}{2} \right] - 2 \right\} \text{ pairs } (A_{2p}, A_{2p+1}) \text{ such that } \left[\frac{k}{2} \right]$$

$$+ 1 \leqslant 2p \leqslant 2p + 1 \leqslant k.$$

If q be the number of such pairs, then

$$q \geqslant \tfrac{1}{2}\left(k - \left[\frac{k}{2}\right] - 2\right)$$

$$\leqslant \tfrac{1}{8}k \text{ (for large } k)$$

$$\leqslant \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)} \text{ (by (4)).} \qquad \ldots(16)$$

Let $\eta = \sum_q \eta_m$, then for $0 < \epsilon_1 < \delta_m$, by Chebysheff's inequality

$$P \mid \eta - E(\eta) \mid \geqslant q\,\epsilon_1 \leqslant \frac{V(\eta)}{q^2 \epsilon_1^2} \leqslant \frac{1}{q\,\epsilon_1^2}$$

for large n, take

$$\epsilon_1 > \frac{1}{\left\{\log\left(\left(\frac{k_n}{t_n}\right)\log n)^{5/\alpha}\right)\right\}^{1/4}} .$$

Then we have,

$$P\{\mid \eta - E(\eta) \mid > q\,\epsilon_1\} \leqslant \frac{32\left\{\log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)\right\}^{3|2}}{\log n}$$

(by (16)).

Thus

$$\mid \eta - E(\eta) \mid < q\,\epsilon_1 \text{ on a set } S_1 \text{ for which}$$

$$P(S_1) \leqslant \frac{32\,e\left\{\log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{3/\alpha}\right)\right\}^{3/2}}{\log n} . \qquad \ldots(17)$$

Again $\eta > E(\eta) - q\,\epsilon_1 \geqslant q\,(\theta - \epsilon_1)$.

Taking $\delta > 2\,\epsilon_1$ for large n, we have

$$\eta > q\,\epsilon_1 > \frac{\log n}{32\left\{\log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)\right\}^{5/4}}$$

$$> \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)} .$$

Hence

$$N_n \geqslant \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)} . \qquad \ldots(18)$$

Now if S denotes the entire exceptional set

Then

$$P(S) \leqslant P(G) + P(S_1)$$

$$< \frac{161\, e \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}{\log n} \quad .. (19)$$

Therefore from (18) and (19), we have

$$P\left\{\omega: N_n(w) < \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}\right\}$$

$$< \frac{161\, e \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}{\log n}.$$

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