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A NOTE ON PRIMARY DECOMPOSITION IN NOETHERIAN NEAR-RINGS

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Hans H. Storrer[3] has introduced the notion of critical left ideals in rings and has given a generalisation of primary decomposition to non-commutative rings. In this paper critcal left ideals in Near-rings are introduced and the relation between critical left ideals and left ideals of type O is studied. Further primary decomposition is also exhibited in modules over near-ring under some conditions.

1. INTRODUCTION

All near-rings are assumed to be zero-symmetric right near-rings with identity. Throughout this paper near-ring under consideration is denoted by K. The definitions of K-module, K-subgroups and submodules are as given in Pilz[2]. However for the sake of continuity the definitions are given again.

Definition 1.1—Let $(M, +)$ be a group and K be a near-ring such that there exists a mapping $u : K \times M \to M$ satisfying the conditions.

$$(k + k')\, m = k\, m + k'\, m$$

$$(kk')\, m = k\, (k'\, m).$$

$1.m = m$ for all $k, k' \in K$, $m \in M$ and 1 is the identify of K. Then $(M, +, u)$ is called a K-module

Definition 1.2—A subgroup N of K-module M is said to be a K-subgroup of M if $(N, +)$ is a subgroup with $KN \subseteq N$.

Definition 1.3—A normal subgroup N of M is called a submodule of M if $k\,(m + n) - k\,m \in N$. For all $m \in M$, $n \in N$ and $k \in K$.

The concepts of essential extension and rational extension were introduced by Barua[1] for near-rings. Here we give a variant of the above notions similar to modules over a ring.

Definition 1.4—Let M be a module over a near-ring K. M is said to be (a) a 'module essential extension' of a non-zero K-subgroup N if for every non-zero submodule N' of M, $N \cap N \neq 0$; (b) an 'essential extension' of a non-zero K-subgroup N if for every non-zero k-subgroup N', $N \cap N' \neq 0$.

Clearly essential extension implies module essential extension. The converse need not be true.

Example—Let $K = \{0, a, b, c\}$ be the klien group under addition and the multiplication be defined by the following table.

·	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then $M = K$ is module essential extension of $N = \{0, a\}$ but not an essential extension of N.

Definition 1.5 : *Rational Extension*—Let N be a non-zero K-subgroup of a K-module M. M is said to be a rational extension of N if the following condition is satisfied :

$N \subseteq A \subseteq M$, A is a K-subgroup of M and $f : A \rightarrow M$ is a

K-homomorphism such that $\ker f \supset N$ implies $f = 0$.

In modules over rings it is known that a rational extension implies essential extension. This need not be the case in case of near-ring modules as seen from the following example.

Example : The dihedral group $D_8 = \{0, a, 2a, 3a, b, a+b, 2a+b, 3a+b\}$. With the addition and multiplication operations defined below is a near-ring with identity a. It is a rational extension of each of its non-zero K-subgroups but not an essential extension of its non-zero K-subgroups.

+	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	a	2a	3a	b	a+b	2a+b	3a+b
a	a	2a	3a	0	a+b	2a+b	3a+b	b
2a	2a	3a	0	a	2a+b	3a+b	b	a+b
3a	3a	0	a	2a	3a+b	b	a+b	2a+b
b	b	3a+b	2a+b	a+b	0	3a	2a	a
a+b	a+b	b	3a+b	2a+b	a	0	3a	2a
2a+b	2a+b	a+b	b	3a+b	2a	a	0	3a
3a+b	3a+b	2a+b	a+b	b	3a	2a	a	0

.	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
$2a$	0	$2a$	0	$2a$	0	0	0	0
$3a$	0	$3a$	$2a$	a	b	$a+b$	$2b+b$	$3a+b$
b	0	b	$2a$	b	b	$a+b$	$2a+b$	$3a+b$
$a+b$	0	$a+b$	0	$3a+b$	0	0	0	0
$2a+b$	0	$2a+b$	$2a$	$2a+b$	b	$a+b$	$2a+b$	$3a+b$
$3a+b$	0	$3a+b$	0	$a+b$	0	0	0	0

Definition 1.6—A K-module M is said to be (i) 'Uniform' if it is an essential extension of each of its non-zero K-subgroups; and

(ii) 'Strongly uniform' if it is uniform and is a rational extension of each of its non-zero K-subgroups.

Notation : If X is a subset of a K-module M then $<X>$ stands for the submodule of M generated by X.

We assume that K-module M satisfies the property (p) :

"$< N_1 \cap N_2 > = < N_1 > \cap < N_2 >$ for any two K-subgroups N_1 and N_2 of M".

Any near-ring K in which every K-subgroup is a submodule of K satisfies this property.

Proposition 1.7—Let M be a K-module with ascending chain condition (A.C.C.) on K-subgroups and satisfying the property (p). Then M has a submodule which is uniform.

PROOF : Suppose M has no submodule which is uniform. Then there exists non-zero K-subgroups N_1 and N_1' such that $N_1 \cap N_1' = 0$. Therefore $M_1 \cap M_1' = 0$ where $M_1 = < N_1 >$ and $M_1' = < N_1' >$. By hypothesis M_1 is not uniform. So there exists non-zero K-subgroups N_2 and N_2' of M_1 such that $N_2 \cap N_2' = 0$. Therefore $M_2 \cap M_2' = 0$ where $< N_2 > = M_2$ and $< N_2' > = M_2'$. Repeating the argument, we get a sequence $\{M_n\}$ of submodules of M which are not uniform such that $M_1 \subset M_1 + M_2 \subset M_1 + M_2 + M_3 \subset \ldots$ contradicting the A.C.C. on K-subgroups.

Hence M has a submodule which is uniform.

2. CRITICAL LEFT IDEALS

Following the notion of critical left ideal as given by Storrer[3] the concept of critical left ideals in near-rings is introduced.

Definition 2.1—A left ideal P of a near-ring K is called a critical left ideal if $M = K/P$ is a strongly uniform K-module.

Definition 2.2—(*a*) A left ideal L of K is of type 2 if $M = K/L$ is a K-module which has no nontrivial K-subgroups.

(b) A left ideal L of K is of type O if $M = K/L$ is a K-module which has no nontrivial submodules.

From the definition it follows that every left ideal of type 2 is a critical left ideal. However we can still strengthen the result as follows.

Proposition 2.3—Every left ideal of type O of K is a critical left ideal.

PROOF : Let L be any left ideal of type O of K. Suppose $\Delta_i = L_i/L$, $i = 1, 2$ are two non-zero K-subgroups of $M = K/L$. Assume that $L_1 \not\subset L_2$ and $L_2 \not\subset L_1$. Choose $r_1 \in L_1 \setminus L_2$ and $r_2 \in L_2 \setminus L_1$. Since L is a modular left ideal of type O, $(K : L) = P$ is a O-promitive ideal of K and $M = K/L$ is a K/P-module of type O.

By density theorem (Pilz[2], 115) there exists $n + p \in N/P$ such that $(n + p)(r_2 + L) = r_1 + L$ which implies that $nr_2 - r_1 \in L \subset L_1$. Consequently $nr_2 \in L_1 \cap L_2$ and $nr_2 \notin L$. Thus $nr_2 + L \in \Delta_1 \cap \Delta_2$. This shows that intersection of any two non-zero K-subgroups of K/L is non-zero. That is $M = K/L$ is uniform.

Now to establish that $M = M/L$ is a rational extension of each of its non-zero K-subgroups, assume that there exists two non-zero K-subgroups Δ_1 and Δ_2 of M such that $\Delta_1 \subset \Delta_2$ and a K-homomorphism $f: \Delta_2 \to M$ with $\ker f \supset \Delta_1$. As before suppose $\Delta_i = L_i/L$, $i = 1, 2$. If $L_1 = L_2$, then clearly $f = 0$. Otherwise choose any $r_2 \in L_2 \setminus L_1$ and $r_1 \in L_1 \setminus L$. Considering $M = K/L$ as a K/P-module and appealing to density theorem again, there exists $n + p \in N/P$ such that $(n + p)(r_1 + L) = r_2 + L$ which implies $nr_1 - r_2 \in L$. Since f is a K-homomorphism and $nr_1 + L \in \Delta_1$ we have $f(r_2 + L) = f(nr_1 + L) = 0$.

This implies that $f = 0$ and M in a rational extension of Δ_1. Hence L is a critical left ideal of K.

In the class of all critical left ideals of a near-ring K, we introduce a relation as follows.

Definition 2.4—If I and J are two left ideals of K, I is said to be related to J if there exists $a \notin I$, $b \notin J$ such that

$$Ia^{-1} = Jb^{-1} \text{ where } Ia^{-1} = \{r \in K / ra \in I\} \text{ and }$$
$$Ja^{-1} = \{r \in K / rb \in J\}.$$

Lemma 2.5—Let K be a near-ring and P and Q be two critical left ideals of K then the following statements are equivalent :

(i) P is related to Q.

(ii) A non-zero K-subgroup of K/P is isomorphic to a non-zero K-subgroup of K/Q.

PROOF : Suppose P and Q are related, then there exists a and b such that

$$Pa^{-1} = Qb^{-1} \text{ and } a \notin P,\ b \notin Q.$$

Then Pa^{-1} is a left ideal of K and K/Pa^{-1} is a non-zero K-module Define :

$$\psi : K/Pa^{-1} \to K/P$$

by

$$\psi\,(x + Pa^{-1}) = xa + P.$$

The map ψ is an isomorphism of K/Pa^{-1} onto a K-subgroup L/P of K/P. Similarly K/Qb^{-1} is isomorphic to a K-subgroup L'/Q of K/Q showing that $L/P \simeq L'/Q$ if $Pa^{-1} = Qb^{-1}$.

Conversely suppose a non-zero K-subgroup N/P is isomorphic to a non-zero K-subgroup M/Q of K/Q.

Let ϕ be a K-isomorphism from N/P to M/Q.

Since $\phi \neq 0$, there exists $\phi\,(a + P) = b + Q \neq 0$.

$$\begin{aligned} \phi\,[r\,(a + p)] &= \phi\,(ra + P) \\ &= r\,\phi\,(a + P) \\ &= r\,(b + Q) = rb + Q. \end{aligned}$$

Since ϕ is mono, $ra \in P \Leftrightarrow rb \in Q$ that is $Pa^{-1} = Qb^{-1}$.

Proposition 2.6—The relation P is related to Q is an equivalence relation in the class of all critical left ideals.

PROOF : The only condition to be verified is the following : If I, J, L are critical left ideals such that I is related to J and J is related to L then I is related to L.

By Lemma 2.4, there exists non-zero K-subgroups M_1/I, N_1/J of K/I and K/J respectively such that $M_1/I \simeq N_1/J$. Similarly there exists non-zero K-subgroups N_2/J and M_2/L of K/J and K/L respectively such that $N_2/J \simeq M_2/L$. Since K/J is uniform, we have $N_1 \cap N_2 \supsetneq J$.

Put $N = N_1 \cap N_2$. Then N/J is isomorphic to a non-zero K-subgroups of K/I and also N/J is isomorphic to a non-zero K-subgroup of K/L. Hence a non-zero K-subgroup of K/I is isomorphic to non-zero K-subgroup of K/L. Hence I is related to L.

The equivalence class containing P is denoted by $[P]$.

3. Associated Left Ideals of a Module M

Definition 3.1—A critical left ideal P is said to belong to M if there exists $0 \neq x \in M$ such that Ann $(x) = P$.

Here we identify, the critical left ideals related to P and say $[P]$ is associated to M. With this identification, the set of all critical ideals belonging to M is denoted by Ass M.

Theorem 3.2—If M satisfies A.C.C. on K-subgroups and also satisfies property (p). Then there exists a non-zero K-subgroup B of M which is strongly uniform.

Proof : By Proposition 1.7, there exists a non-zero K-subgroup N of M which is uniform.

If N is strongly uniform, nothing to prove. Suppose now N is not strongly uniform. But by our choice N is uniform.

Therefore N cannot be a rational extension of each of its non-zero K-subgroups.

That is, there exists a maximal K-subgroup A of N such that N is not a rational extension of A (since M satisfies A.C.C. on K-subgroups). Hence there exists K-subgroup N' such that $A \underset{\neq}{\subset} N' \subset N$, and $\phi : N' \to N$, a non-zero homomorphism with Ker $\phi \supset A$.

Let $\phi (N') = B \neq 0$.

Then B is a non-zero K-subgroup of N. We claim that B is a rational extension of each of its non-zero K-subgroups.

Let $B' \neq 0$ be a K-subgroup of B.

Consider $\phi^{-1} (B') = A'$ then $A \underset{\neq}{\subset} A' \subset N$ and N is a rational extension of A' (by the maximality of A).

Let $f : B'' \to B$ be a homomorphism where $B' \subset B'' \subset B$ with Ker $f \supset B'$.

If $\phi^{-1} (B'') = A''$ then $\phi^{-1} (B) \supset \phi^{-1} (B'') \supset \phi^{-1} (B') \supset A'$.

Then $f \circ \phi : A'' \to N$ is a homomorphism such that Ker $f \circ \phi \supset A'$. But N is a rational extension of A'. Therefore $f \circ \phi = 0$ and hence $f = \circ$ on B''.

That is, B is a rational extension of B'.

Therefore $B \subset N$ and B is rational extension of each of its K-subgroup. As N is uniform, B is also uniform. Therefore B is strongly uniform.

As a corrollory to the above, we have :

Corollary 3.3—If M satisfies A C.C. on K-subgroups and also property (p), then Ass $M \neq \Phi$.

PROOF : By Theorem 3.2, M has a K-subgroup N which is strongly uniform. Let $0 \neq x \in N$. Then $Kx \cong K/P$ where $P = \text{Ann}(x)$ and Kx is strongly uniform. Hence P is a critical left ideal associated to M and $\text{Ass}(M) \neq \Phi$. Some of the properties are the following :

Proposition 3.4—Let M be a K-module satisfying A.C.C. on K-subgroups.

(a) If M is the union of submodules of Mi, then $\text{Ass } M = U_i \text{ Ass } M_i$.

(b) If P is a critical left ideal then $\text{Ass}(K/P) = \{[P]\}$.

(c) If $N \subseteq M$, then $\text{Ass } N \subseteq \text{Ass } M \subseteq \text{Ass } N \cup \text{Ass } M/N$.

(d) If M is the direct sum of submodules M_i, then $\text{Ass } M = U_i \text{ Ass } M_i$

PROOF : (b) To show $\text{Ass}(K/P) = \{[P]\}$.

Consider $\text{Ann}(1 + P) = [r \in K/r(1 + P) = \bar{0}) = P$.

If there exists $x \in K$ such that $\text{Ann}(x + P) = Q$.

$$Q.1^{-1} = \text{Ann}(x + P) = [r \in K/r(x + P) = 0] = Px^{-1}.$$

Hence P is related to Q.

Therefore $\text{Ass}(K/P) = \{[P]\}$.

(c) If $N \subseteq M$, then $\text{Ass}(N) \subseteq \text{Ass}(M)$ is clear.

Let N be a submodule of M.

Let $P \in \text{Ass } M$ and $P = \text{Ann}(x)$, $0 \neq x \in M$.

Case (*i*)—If $Kx \cap N = <0>$

$$\begin{aligned} \text{Ann}(x + N) \text{ in } M/N &= [y \in K/ y(x + N) = 0 + N] \\ &= [y \in K/yx = 0] \\ &= \text{Ann}(x) \text{ in } M. \end{aligned}$$

That is $P \in \text{Ass}(M/N)$.

Case (*ii*)—If $Kx \cap N \neq <0>$. Let $N' = Kx \cap N$ consider N' as K-module.

Then $\text{Ass}(N') \neq \Phi$ by corollory 3.3.

Let $Q \in \text{Ass}(N') \subset \text{Ass } N$.

$Q \in \text{Ass}(N') \Rightarrow$ There exists $y \in N'$ such that $\text{Ann}(y) = Q$

$y \in Kx \Rightarrow$ There exists $a \in K$ such that $y = ax$.

$$\begin{aligned} Q = \text{Ann}(y) &= \{z/zy = 0\} \\ &= \{z/(za)x = 0\} \\ &= \{z/za \in P\} = Pa^{-1}. \end{aligned}$$

So $Pa^{-1} = Q1^{-1}$ and P is related to Q.

Hence $[P] \in$ Ass (N).

4. Primary Submodules

A K-module M is said to be coprimary if Ass M consists of a single element and and a submodule N of M is said to be primary if quotient module M/N is coprimary.

Theorem 4.1—Suppose M is a module which satisfies A.C.C. on K-subgroups and uniform. Then M is coprimary.

Proof : Let $P \in$ Ass M, $P =$ Ann (x), $Kx \simeq K/P$.

If $Q \in$ Ass M, then $Q =$ Ann (y), $Ky \simeq K/Q$.

Since M is uniform, $Kx \cap Ky \neq 0$.

Let $C = Kx \cap Ky$. Then C is isomorphic to a non-zero K-subgroup of K/P and also a non-zero K-subgroup of K/Q.

Therefore P is related to Q.

$Q \sim P \in$ Ass M.

Therefore Ass $M = \{[P]\}$.

Definition 4.2—Let N be a submodule of M. N is said to be irreducible if for any two submodules N_1 and N_2, $N \underset{\neq}{\subset} N_1$, $N \underset{\neq}{\subset} N_2 \Rightarrow N \underset{\neq}{\subset} N_1 \cap N_2$.

And N is said to be 'strongly irreducible' if for any two K-subgroups N_1 and N_2, $N \underset{\neq}{\subset} N_1$ and $N \underset{\neq}{\subset} N_2 \Rightarrow N \underset{\neq}{\subset} N_1 \cap N_2$.

If M satisfies property (P), clearly a submodule N is irreducible if and only if it is strongly irreducible.

Hence if N is an irreducible submodule then M/N is uniform.

Theorem 4.3—If M is a K-module which satisfies A.C.C. on K-subgroups then every submodule of M is a finite intersection of irreducible submodules.

This can be proved by using the A.C.C. on K-subgroups.

Definition 4.4—A primary decomposition of a submodule N of M is a representation of N as a finite intersection of primary submodules.

Theorem 4.5—If M is a K-module satisfying A.C.C. on K-subgroups and property (p), then every submodule N of M has a primary decomposition.

PROOF : By Theorem 4.3, a submodule N of M can be written as $N = N_1 \cap N_2 \cap \ldots \cap N_k$ where each N_i is irreducible submodule. By the property (p) each N_i is strongly irreducible.

Then for each i, M/N_i is uniform and satisfies A.C.C. on K-subgroups and it is coprimary by Theorem 4.1. Thus, N_i is primary.

5. Z-S-COPRIMARY SUBMODULES

It can be shown that every primary decomposition has reduced primary decomposition and uniqueness by the familiar methods.

Lemma 5.1—Let K be a near-ring with 1 satisfying the conditions.

(a) Possesses A.C.C. on ideals.

(b) Every left K-subgroup is an ideal of K.

Then every critical left ideal of K is a prime ideals.

PROOF : By condition (b), P is an ideal of K and by definition of critical left ideal, K/P is strongly uniform.

If P is not prime, there exists ideals A and B such that $P \underset{\neq}{\subset} A$, $P \underset{\neq}{\subset} B$ and $BA \subset P$. Since $P \neq A$, there exists $a \in A$ and $a \notin P$.

Define mapping f : $K/P \to K/P$ as follows :

$$f(x + P) = xa + P$$

clearly this is a module homomorphism and Ker $f \supset B/P$ and $f \neq 0$.

For if $f = 0$, then $xa \in P$ for every $x \in K$.

That is $Ka \subset P$ which is not the case since $a \in Ka$ and $a \notin P$.

This is a contradiction to the hypothesis that K/P is a rational extension of each of its non-zero K-subgroups.

Hence P is prime.

Definition 5.2—An ideal Q of K is said to be Z-S-coprimary if $Px \subset Q \Rightarrow P \subset r(Q)$ or $x \in Q$

where $r(Q)$ is the intersection of prime ideals.

Theorem 5.3—If K-satisfies conditions above, then an ideal Q which is Z-S-coprimary is coprimary in the sense of this article.

PROOF : $P \in$ Ass (K/Q) and $P =$ Ann $(x + Q)$.

That is $Px \subset Q$, $x \notin Q$.

Therefore $P \subset r(Q)$ but $r(Q) \subset P$ (since P is prime)

therefore $P = r(Q)$ which implies Ass $(K/Q) = \{[P]\}$.

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